I YEAR - I SEMESTER
COURSE CODE: 7BMA1C2

CORE COURSE - II – ALGEBRA AND TRIGONOMETRY

Unit – I

Summation of Series – Binomial Series – Exponential Series – Logarithmic Series.

Unit – II

Relation between roots and coefficients – Sum of the powers of the roots – Reciprocal Equation – Transformation of Equations.

Unit – III

Multiple Roots – Nature and position of roots –Descarte's rule of Signs, Rolle's theorem – Sturm's functions – Problems – Finding number and position of the real roots – Finding the nature and position of the roots (Cardans&Ferrar's method not included) – Approximate solution of Numerical equations – Newton's method – Horner's method.

Unit – IV

Applications of Demoivre's Theorem – Expression for $\sin n\theta$, $\cos n\theta$, $\tan n\theta$. Expression for $\sin^n\theta$, $\cos^n\theta$ - Expansion of $\sin\theta$, $\cos\theta$, $\tan\theta$ in powers of θ.

Unit – V

Hyperbolic functions – Inverse hyperbolic functions, and logarithm of a complex number.

Text Books:

1. Summation of Series and Trigonometry by Dr.S.Arumugam and A.Thangapandi Isaac – New Gamma Publishing House,Palayamkottai.
2. Theory of Equations, Theory of Numbers and Trigonometry by Dr. S.Arumugam and A.ThangapandiIssac – New Gamma Publishing House, Palayamkottai July 2011.

Unit I	Chapter 1 sections 1.1 – 1.3 of (1)
Unit II	Chapter 5 sections 5.2 to 5.5 of (2)
Unit III	Chapter 5 sections 5.6, 5.7, 5.10 of (2)
Unit IV	Chapter 6 of(2)
Unit V	Chapter 7 and Chapter 8 of (2)

Books for Reference:

1. Trigonometry by S.Narayanan, T.K.ManicavachagomPillay.
2. Algebra Volume – I by T.K.ManicavachagomPillay, T.Natarajan, KS.Ganapathy.

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Summation of series:

Binomial series:

When n is a positive integer $(x+a)^n$ can be expanded as

$$(x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \dots + nc_r x^{n-r} a^r + \dots + a^n$$

This known as the binomial theorem for the positive integer n.

When n is a rational number $(1+x)^n$ can be expanded as an infinite series when $-1 < x < 1$ (ie) $|x| < 1$ and it is given by

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots$$

This is known as binomial series for $(1+x)^n$ where n is a rational number.

Special cases:

Result 1

$$(1+x)^{-1} = 1 - \frac{x}{1!} + \frac{-1(-1-1)x^2}{2!} - \frac{-1(-2)(-3)}{3!}x^3 + \dots$$

$$= 1 - x + \frac{2x^2}{2\times1} - \frac{6x^3}{3\times2\times1}$$

$$= 1 - x + x^2 - x^3$$

$$= \sum_{r=0}^{\infty} (-1)^r x^r$$

Result II

$$(1-x)^{-1} = 1+x+x^2+x^3 = \sum_{r=0}^{\infty} x^r$$

Result III

$$(1+x)^{-2} = 1+\frac{(-2)x}{1!}+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}x^3$$

$$= 1-2x+\frac{6x^2}{2!}-\frac{24x^3}{3!}$$

$$= 1-2x+3x^2-4x^3$$

$$= \sum_{r=0}^{\infty} (-1)^r (r+1) x^r$$

Result IV

$$(1-x)^{-2} = 1+2x+3x^2+4x^3$$

$$= \sum_{r=0}^{\infty} (r+1)x^r$$

Result V

$$(1+x)^{-3} = 1-3x+\frac{(-3)(-3-1)}{2!}x^2+\frac{(-3)(-3-1)(-3-2)}{3!}x^3$$

$$= 1-3x+\frac{3\times 4}{2!}x^2+\frac{-3\times 4\times 5}{3\times 2!}x^3$$

$$= 1-3x+\frac{3\times 4}{2!}x^2-\frac{4\times 5}{2!}x^3$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(r+1)(r+2)}{2!} x^r$$

Result VI

$$(1-x)^{-3} = 1+3x+\frac{3\times4}{2!}x^2+\frac{4\times5}{2!}x^3$$

$$= \sum_{r=0}^{\infty}\frac{(r+1)(r+2)}{2!}x^r$$

Result VII

$$(1-x)^{-p/q} = 1-\frac{\frac{p}{q}(-x)}{1!}+\frac{\left(\frac{-p}{q}\right)\left(\frac{-p}{q}-1\right)(-x)^2}{2!}+\frac{\left(\frac{-p}{q}\right)\left(\frac{-p}{q}-1\right)\left(\frac{-p}{q}-2\right)(-x)^3}{3!}+\cdots$$

$$= 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\cdots$$

Problems :

1. Find the general term in the expansion $(4-7x)^{-\frac{2}{5}}$ stating when will the expansion be valid.

Solution :

$$(4-7x)^{-2/5} = (4)^{-2/5}\left(1-\frac{7x}{4}\right)^{-2/5} = 2^{-4/5}\left(1-\frac{7x}{4}\right)^{-2/5}$$

$\left(1-\frac{7x}{4}\right)^{-2/5}$ can be expanded in binomial series if

$\left|\frac{7x}{4}\right|<1$ (i.e) if $|x|<\frac{4}{7}$

The general term T_{r+1} in $\left(1-\frac{7x}{4}\right)^{-2/5}$ is

$$\left(1-\frac{7x}{4}\right)^{-2/5} = \frac{\frac{-2}{5}\left(\frac{-2}{5}-1\right)\left(\frac{-2}{5}-2\right)\cdots\cdots\left(\frac{-2}{5}-r+1\right)\left(\frac{-7x}{4}\right)^r}{r!}$$

$$= \frac{-\frac{2}{5}\left(\frac{-7}{5}\right)\left(\frac{-12}{5}\right)\cdots\cdots\left(-\frac{5r+3}{5}\right)\left(\frac{-7x}{4}\right)^r(-1)^r}{r!}$$

$$= \frac{-2(-7)(-12)\cdots(-5r+3)\left(-\frac{7x}{4}\right)^r(-1)^r}{5^r\, r!}$$

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$$= \frac{(-1)^r\, 2(7)(12)\cdots(5r-3)\,(-1)^r\left(\frac{7x}{20}\right)^r}{r!}$$

$$= \frac{2(7)(12)\cdots(5r-3)(-1)^{2r}\left(\frac{7x}{20}\right)^r}{r!}$$

$$= \frac{2(7)(12)\cdots(5r-3)}{r!}\left(\frac{7x}{20}\right)^r$$

Hence the general term $= (2)^{-4/5}\left(1-\frac{7x}{4}\right)^{-2/5}$

$$= (2)^{-4/5}\, \frac{2(7)(12)\cdots(5r-3)}{r!}\left(\frac{7x}{20}\right)^r$$

2. If $|x| < \frac{1}{2}$ prove that co-efficient of x^n in the expansion of $(2-4x)(1-2x)^{-2}$ is $\boxed{2^{n+1}}$

Solution:

$$|x| < \tfrac{1}{2} \Rightarrow |2x| < 1$$

$\left((1-x)^{-2} = \sum_{r=0}^{\infty} (r+1)x^r \right)$

Hence we can expanded $(1-2x)^{-2}$ in binomial series is $\sum_{r=0}^{\infty}(r+1)(2x)^r$ by Result-4

Now, $(2-4x)(1-2x)^{-2} = (2-4x)\sum_{r=0}^{\infty}(r+1)(2x)^r$

$$= (2-4x)\left[1+2(2x)+3(2x)^2+\cdots+n(2x)^{n-1}+(n+1)(2x)^n+\cdots\right]$$

$$= (2-4x)\left[1+2(2x)+3(2^2)x^2+\cdots+n(2^{n-1})x^{n-1}+(n+1)2^n x^n\right]$$

$$= (2-4x)\left[n.2^{n-1}.x^{n-1} + (n+1)2^n x^n\right]$$

$$= 2(n+1)2^n - 4(n.2^{n-1})$$

$$= (2n+2)2^n - 2^2(n.2^{n-1})$$

$$= 2n.2^n + 2^{n+1} - n.2^{n+1}$$

$$= n.2^{n+1} + 2^{n+1} - n.2^{n+1}$$

$$= 2^{n+1}$$

3. Find the coefficient of x^n in the expansion $(1-2x+3x^2-4x^3)^{-n}$

solution :

$$(1-2x+3x^2-4x^3)^{-n} = \left[(1+x)^{-2}\right]^{-n} = (1+x)^{2n}$$

coefficient of x^n in $(1-2x+3x^2-4x^3+\cdots)^{-n}$ is same as the coefficient of x^n in $(1+x)^{2n}$ and it is

$$= \frac{2n(2n-1)(2n-2)\cdots(2n-(n-1))}{n!}$$

$n(n-1)(n-2)\cdots(n-n+1)$

$2n(2n-1)(2n-2)\cdots(2n-n+1)$

$$= \frac{2n(2n-1)(2n-2)\cdots(n+1)}{n!}$$

$$= \frac{2n(2n-1)\cdots(n+1)[n(n-1)\cdots 2.1]}{n!\,[n(n-1)\cdots 2.1]}$$

$$= \frac{2n!}{n!\,n!}$$

4. Express $\frac{1}{1-x-6x^2}$ as the sum of partial fractions and hence show that

$$1 + \frac{(n-1)}{1!}.6 + \frac{(n-2)(n-3)}{2!}.6^2 + \cdots = \frac{1}{5}\left[3^{n+1} + (-1)^n 2^{n+1}\right]$$

Solution:

$$\frac{1}{1-x-6x^2} = \frac{1}{(1-3x)(1+2x)}$$

$-6x^2 - x + 1$ $\quad$ 6

$$\begin{array}{c|c} -6 & -1 \\ \hline -3 & 2 \\ -6x & -6x \end{array}$$

$(2x+1)(1-3x)$

$$= \frac{A}{(1-3x)} + \frac{B}{(1+2x)}$$

$$\frac{1}{1-x-6x^2} = \frac{A(1+2x)+B(1-3x)}{(1-3x)(1+2x)}$$

$$1 = A(1+2x)+B(1-3x)$$

$$x = \frac{-1}{2} \Rightarrow \quad 1 = A\left[1+2\left(\frac{-1}{2}\right)\right] + B\left[1-3\left(\frac{-1}{2}\right)\right]$$

$$1 = 0 + B\left(1+\frac{3}{2}\right)$$

$$1 = \frac{5B}{2} \Rightarrow B = \frac{2}{5}$$

$$x = \frac{1}{3} \Rightarrow \quad 1 = A\left[1+2\left(\frac{1}{3}\right)\right] + B\left[1-3\left(\frac{1}{3}\right)\right]$$

$$1 = A\left[1+\frac{2}{3}\right] + 0$$

$$1 = \frac{5A}{3} \Rightarrow A = \frac{3}{5}$$

$$\frac{1}{1-x-6x^2} = \frac{3}{5(1-3x)} + \frac{2}{5(1+2x)}$$

$$= \frac{3}{5}(1-3x)^{-1} + \frac{2}{5}(1+2x)^{-1}$$

Expanding $(1-3x)^{-1}$ and $(1+2x)^{-1}$ in the binomial series for $|x| < \frac{1}{3}$ and $|x| < \frac{1}{2}$ (i.e) for $|x| < \frac{1}{2}$, we get the coefficient of x^n for the given expression as

$$= \left(\frac{3}{5}\right)3^n + \left(\frac{2}{5}\right)(-2)^n$$

coefficient of x^n in $\frac{1}{1-x-6x^2} = \frac{1}{5}\left[3^{n+1}+(-1)^n 2^{n+1}\right] \longrightarrow (1)$

$$\frac{1}{1-x-6x^2} = \frac{1}{[1-x(1+6x)]}$$

$$= [1-x(1+6x)]^{-1}$$

$$= 1+x(1+6x)+x^2(1+6x)^2+\dots+x^n(1+6x)^n+\dots$$

coefficient of x^n in $x^n(1+6x)^n = 1$

coefficient of x^n in $x^{n-1}(1+6x)^{n-1} = \frac{(n-1)}{1!}\cdot 6$

coefficient of x^n in $x^{n-2}(1+6x)^{n-2} = \frac{(n-2)(n-3)}{2!}\cdot 6^2$

$\therefore$ coefficient of x^n in

$$\frac{1}{1-x-6x^2} = 1+\frac{(n-1)}{1!}\cdot 6+\frac{(n-2)(n-3)}{2!}\cdot 6^2+\dots \longrightarrow (2)$$

Hence from (1) and (2) we get the required result.

5. Find the general term in the expansion $(1-2x)^{-2}$

Solution:

$$(1-2x)^{-2} = \frac{-2(-2-1)(-2-2)\dots(-2-r+1)(-2x)^r}{r!}$$

$$= \frac{-2(-3)(-4)\dots(-1-r)(-2x)^r(-1)^r}{r!}$$

$$= \frac{2(3)(4)\dots(1+r)(2x)^r(-1)^r(-1)^r}{r!}$$

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$$= \frac{2(3)(4)\cdots(1+r)(2x)^r(-1)^{2r}}{r!}$$

$$= \frac{2(3)(4)\cdots(1+r)(2x)^r}{r!}$$

Problem 5:

Find the coefficient of x^n when $\frac{7+x}{(1+x)(1+x^2)}$ expanded in ascending power of x.

solution:

$$\frac{7+x}{(1+x)(1+x^2)} = \frac{A}{(1+x)} + \frac{Bx+C}{(1+x^2)}$$

$$= \frac{A(1+x^2)+[Bx+C](1+x)}{(1+x)(1+x^2)}$$

$$7+x = A(1+x^2)+[Bx+C](1+x)$$

$x=0 \Rightarrow$ $7+0 = A(1+0)+[0+C](1+0)$

$$7 = A+C$$

$x=-1 \Rightarrow$ $7-1 = A(1+1)+[-B+C](0)$

$$6 = 2A+0$$

$$\boxed{A=3}$$

$7 = 3+C$

$7-3 = C$

$\boxed{C=4}$

$x=1 \Rightarrow$ $8 = 3(1+1)+[B+4](2)$

$8 = 6+2B+8$

$2B = -6$

$\boxed{B=-3}$

$$\therefore \frac{7+x}{(1+x)(1+x^2)} = \frac{3}{(1+x)} + \frac{4-3x}{(1+x^2)}$$

$$= 3(1+x)^{-1}+[4-3x](1+x^2)^{-1}$$

$$= 3(1-x+x^2-x^3+\cdots) + (4-3x)(1-x^2+x^4-x^6+\cdots)$$

case (i) r is an odd integer say $r = 2n+1, n \in N$

$\therefore$ coefficient of

$$x^r = \text{coefficient of } x^{2n+1} \text{ in (1)}$$

$$= -3 + (-3)(-1)^n$$

$$= -3 + (-3)(-1)^{\frac{r-1}{2}}$$

case (ii) r is an even integer say $r = 2n, n \in N$

$\therefore$ coefficient of x^r = coefficient of x^{2n}

$$= -3 + 4(-1)^n$$

$$= -3 + 4(-1)^{r/2}$$

Problem : 6

$$2\sqrt{x^2+4} - \sqrt{4x^2+9} = \begin{cases} 1 - \frac{1}{6}x^2 & \text{when } x \text{ is small} \\ \frac{7}{4x} - \frac{175}{64x^3} & \text{when } x \text{ is large} \end{cases}$$

case (i) x is small

$$2\sqrt{x^2+4} - \sqrt{4x^2+9} = 4\left[1 + \frac{x^2}{4}\right]^{1/2} - 3\left[1 + \frac{4x^2}{9}\right]^{1/2}$$

since x is small $\left|\frac{x^2}{4}\right| < 1$ and $\left|\frac{4x^2}{9}\right| < 1$

$1+nx$

$$2\sqrt{x^2+4} - \sqrt{4x^2+9} = 4\left[1 + \frac{x^2}{8} + \cdots\right] - 3\left[1 + \frac{2x^2}{9} + \cdots\right]$$

$$= 4 + \frac{4x^2}{8} - 3 - \frac{6x^2}{9}$$

$$= 1 + x^2\left[\frac{1}{2} - \frac{2}{3}\right]$$

$$= 1 - \frac{1}{6}x^2$$

case (ii) x is large. Hence $\frac{1}{x}$ is very small

$$2\sqrt{x^2+4}-\sqrt{4x^2+9} = 2x\left[1+\frac{4}{x^2}\right]^{1/2} - 2x\left[1+\frac{9}{4x^2}\right]^{1/2}$$

since $\frac{1}{x}$ is very small $\left|\frac{4}{x^2}\right|<1$ and $\left|\frac{9}{4x^2}\right|<1$

$$\therefore 2\sqrt{x^2+4}-\sqrt{4x^2+9} = 2x\left[1+\frac{2}{x^2}-\frac{2}{x^4}+\cdots\right] - 2x\left[1+\frac{9}{8x^2}-\frac{81}{128x^4}\right]$$

$$= 2x\left[1+\frac{2}{x^2}-\frac{2}{x^4}-1-\frac{9}{8x^2}+\frac{81}{128x^4}\right] \qquad = 2x+\frac{4x}{x^2}-\frac{4x}{x^4}-2x-\frac{18x}{8x^2}+\frac{162x}{128x^4}$$

$$= 2x\left(\frac{1}{x^2}\left[2-\frac{9}{8}\right]+\frac{1}{x^4}\left[\frac{81}{128}-2\right]\right) = \frac{4}{x}-\frac{4}{x^3}-\frac{9}{4x}+\frac{81}{64x^3}$$

$$= 2x\left[\frac{7}{8x^2}+\frac{-175}{128x^4}\right] \qquad = \frac{4}{x}-\frac{9}{4x}+\frac{81}{64x^3}-\frac{4}{x^3}$$

$$= \frac{7}{4x}-\frac{175}{64x^3} \qquad = \frac{1}{x}\left[4-\frac{9}{4}\right]+\frac{1}{x^3}\left[\frac{81}{64}-4\right]$$

$$= \frac{1}{x}\left(\frac{7}{4}\right)+\frac{1}{x^3}\left(-\frac{175}{64}\right)$$

$$= \frac{7}{4x}-\frac{175}{64x^3}$$

Problem : 7

If x is so small that its square and higher powers may be neglected prove that

$$\frac{\sqrt{1+x}\,(4-3x)^{3/2}}{(8+5x)^{1/3}} = 4-\frac{10x}{3} \text{ (nearly)}$$

Solution :

$$\frac{\sqrt{1+x}\,(4-3x)^{3/2}}{(8+5x)^{1/3}} = (1+x)^{1/2}(4-3x)^{3/2}(8+5x)^{-1/3}$$

$$= (1+x)^{1/2}\, 4^{3/2}\left(1-\frac{3x}{4}\right)^{3/2} 8^{-1/3}\left(1+\frac{5x}{8}\right)^{-1/3}$$

$$= 2^{2\times\frac{3}{2}} \cdot 2^{-\frac{1}{3}\times 3}(1+x)^{1/2}\left(1-\frac{3x}{4}\right)^{3/2}\left[1+\frac{5x}{8}\right]^{-1/3}$$

$$= 2^{3-1}(1+x)^{1/2}\left(1-\frac{3x}{4}\right)^{3/2}\left(1+\frac{5x}{8}\right)^{-1/3}$$

$$= 4\left(1+\frac{1}{2}x+\cdots\right)\left(1-\frac{9x}{8}+\cdots\right)\left(1-\frac{5x}{24}+\cdots\right)$$

$$= 4\left[1+x\left(\frac{1}{2}-\frac{9}{8}-\frac{5}{24}\right)\right]$$

$$= 4\left[1+x\left(\frac{12-27-5}{24}\right)\right]$$

$$= 4\left[1+x\left(\frac{-20}{24}\right)\right] = \left[4+x\left(-\frac{10}{3}\right)\right]$$

$$= 4-\frac{10x}{3}$$

Problem : 8 :

calculate $(1.01)^{1/2}-(0.99)^{1/2}$ correct to six places of decimals. Solution :

$$(1.01)^{1/2}-(0.99)^{1/2} = (1+.01)^{1/2}-(1-.01)^{1/2}$$

$1+nx+\frac{n(n-1)}{2!}x^2+$

$$\approx\left[1+\frac{1}{2}(.01)+\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)(.01)^2+\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{1}{6}\right)(.01)^3+\cdots\right]$$

$$-\left[1+\frac{1}{2}(-.01)+\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)(-.01)^2+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{1}{6}\right)(-.01)\right.$$

$$\approx 2\left[\frac{1}{2}(.01)+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{1}{6}\right)(.01)^3+\cdots\right]$$

$$\approx 2\left[\frac{1}{2}(.01)+\frac{1}{16}(.000001)\right]$$

$$\approx 2\left[.005+.00000006\right]$$

$$\approx 2\left[.00500006\right]$$

$\approx .01000012$

≈ 0.010000

Problem : 9 show that

$$1+n\left(\frac{2a}{1+a}\right)+\frac{n(n+1)}{1.2}\left(\frac{2a}{1+a}\right)^2+\cdots=\left(\frac{1+a}{1-a}\right)^n$$

Solution :

Put $\frac{2a}{1+a}=y$

$$LHS = 1+\frac{ny}{1!}+\frac{n(n+1)}{2!}y^2+\cdots\cdots=(1-y)^{-P/q}?$$

by Result 7

$$1+\frac{P}{1!}\left(\frac{x}{q}\right)+\frac{P(P+q)}{2!}\left(\frac{x}{q}\right)^2+\cdots=(1-x)^{-P/q}$$

$P=n,\ q=1$ and $\frac{x}{q}=y \Rightarrow x=y$

$$LHS = (1-y)^{-n}$$

$$=\left[1-\frac{2a}{1+a}\right]^{-n}$$

$$=\left[\frac{1-a}{1+a}\right]^{-n}$$

$$=\left[\frac{1+a}{1-a}\right]^{n}=RHS$$

Problem : 10 Show that

$$1-\frac{n+x}{1+x}+\frac{(n+2x)(n-1)}{2!\,(1+x)^2}-\frac{(n+3x)(n-1)(n-2)}{3!\,(1+x)^3}=0$$

Solution :

$$LHS = \left[1 - \frac{n}{1!}\left(\frac{1}{1+x}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{1+x}\right)^2 - \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{1+x}\right)^3 + \cdots\right]$$

$$+ \left[\frac{-x}{1+x} + \frac{2x(n-1)}{2!\,(1+x)^2} - \frac{3x(n-1)(n-2)}{3!\,(1+x)^3} + \cdots\right]$$

$$= \left[1 - \frac{n}{1!}\left(\frac{1}{1+x}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{1+x}\right)^2 - \cdots\right] - \frac{x}{1+x}\left[1 - \frac{n-1}{1!}\,\frac{1}{1+x} + \frac{(n-1)(n-2)}{2!}\left(\frac{1}{1+x}\right)^2 - \cdots\right]$$

$$= \left[1 - \frac{1}{1+x}\right]^n - \frac{x}{1+x}\left[1 - \frac{1}{1+x}\right]^{n-1}$$

$$\frac{-x}{1+x}\left[1 - \frac{(n-1)}{1!}\,\frac{1}{(1+x)^2} + \frac{(n-1)(n-2)}{2(1+x)^2}\right]$$

$$= \left[\frac{1+x-1}{1+x}\right]^n - \frac{x}{1+x}\left[\frac{1+x-1}{1+x}\right]^{n-1}$$

$$= \left[\frac{x}{1+x}\right]^n - \frac{x}{1+x}\left[\frac{x}{1+x}\right]^{n-1}$$

$$= \left[\frac{x}{1+x}\right]^n - \left[\frac{x}{1+x}\right]^{n+1-1}$$

$$= \left[\frac{x}{1+x}\right]^n - \left[\frac{x}{1+x}\right]^n$$

$$= 0$$

Problem : 11 prove that

$$1 + \frac{2n}{3} + \frac{2n(2n+2)}{3.6} + \frac{2n(2n+2)(2n+4)}{3.6.9} + \cdots$$

$$= 2^n\left[1 + \frac{n}{3} + \frac{n(n+1)}{3.6} + \frac{n(n+1)(n+2)}{3.6.9} + \cdots\right]$$

Solution :

$$LHS = 1 + \frac{n}{1!}\left(\frac{2}{3}\right) + \frac{n(n+1)}{2!}\left(\frac{2}{3}\right)^2 + \cdots$$

by using the result

$$1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \cdots = (1-x)^{-p/q}$$

$$p = n,\ q = 1,\ \frac{x}{q} = \frac{2}{3} \Rightarrow x = 2/3$$

$$= \left[1 - \frac{2}{3}\right]^{-n/1}$$

$$= \left[\frac{3-2}{3}\right]^{-n} = \left(\frac{1}{3}\right)^{-n} = 3^n$$

$$RHS = 2^n\left[1 + \frac{n}{1!}\left(\frac{1}{3}\right) + \frac{n(n+1)}{2!}\left(\frac{1}{3}\right)^2 + \cdots\right]$$

$$= 2^n\left[1 - \frac{1}{3}\right]^{-n}$$

$$= 2^n\left[\frac{2}{3}\right]^{-n} = 2^n \cdot 2^{-n}\left(\frac{1}{3}\right)^{-n} = 3^n$$

$\therefore$ LHS = RHS

Problem : 12 sum to ∞ the series

$$1 + \frac{1}{5} + \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} + \cdots$$

Solution :

$$\text{Let } S = 1 + \frac{1}{5} + \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} + \cdots$$

$$\therefore S = 1 + \frac{1}{1!}\left(\frac{1}{5}\right) + \frac{4.1}{2!}\left(\frac{1}{5}\right)^2 + \frac{1.4.7}{3!}\left(\frac{1}{5}\right)^3$$

by using the result

$$1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\cdots=(1-x)^{-p/q}$$ Formula

$p=1,\ q=3,\ \frac{x}{q}=\frac{1}{5} \Rightarrow x=\frac{3}{5}$

$$\therefore S=\left[1-\frac{3}{5}\right]^{-1/3}$$

$$=\left[\frac{2}{5}\right]^{-1/3}=\left[\frac{5}{2}\right]^{1/3}$$

Problem : 13 sum to ∞ the series

$$\left(\frac{1}{2}\right)^2+\frac{1}{2!}\left(\frac{1}{2}\right)^4+\frac{1\cdot 3}{3!}\left(\frac{1}{2}\right)^6+\cdots\infty$$

solution :

Let $S=\left(\frac{1}{2}\right)^2+\frac{1}{2!}\left(\frac{1}{2}\right)^4+\frac{1\cdot3}{3!}\left(\frac{1}{2}\right)^6+\cdots$

$$\therefore S=\frac{1}{1!}\left(\frac{1}{4}\right)+\frac{1}{2!}\left(\frac{1}{4}\right)^2+\frac{1\cdot3}{3!}\left(\frac{1}{4}\right)^3+\cdots$$

$$-S=\frac{-1}{1!}\left(\frac{1}{4}\right)+\frac{-1\cdot1}{2!}\left(\frac{1}{4}\right)^2+\frac{-1\cdot1\cdot3}{3!}\left(\frac{1}{4}\right)^3+\cdots$$

$$-S+1=1-\frac{1}{1!}\left(\frac{1}{4}\right)+\frac{-1\cdot1}{2!}\left(\frac{1}{4}\right)^2+\frac{-1\cdot1\cdot3}{3!}\left(\frac{1}{4}\right)^3+\cdots$$

$$1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\cdots=(1-x)^{-p/q}$$ Formula

$p=-1,\ q=2,\ \frac{x}{q}=\frac{1}{4} \Rightarrow x=\frac{1}{2}$

$$\therefore -S+1=\left[1-\frac{1}{2}\right]^{1/2}$$

$$=\left(\frac{1}{2}\right)^{1/2}$$

$$=\frac{1}{\sqrt{2}}$$

$$S=1-\frac{1}{\sqrt{2}}$$

Problem: 14 sum to ∞ the series

$$\frac{3}{18} + \frac{3\cdot 7}{18\cdot 24} + \frac{3\cdot 7\cdot 11}{18\cdot 24\cdot 30} + \cdots$$

Solution:

$$\text{Let } S = \frac{3}{18} + \frac{3\cdot 7}{18\cdot 24} + \frac{3\cdot 7\cdot 11}{18\cdot 24\cdot 30} + \cdots$$

$$= \frac{3}{3}\left(\frac{1}{6}\right) + \frac{3\cdot 7}{3\cdot 4}\left(\frac{1}{6}\right)^2 + \frac{3\cdot 7\cdot 11}{3\cdot 4\cdot 5}\left(\frac{1}{6}\right)^3 + \cdots$$

$$\therefore \frac{S(-5)(-1)}{1\cdot 2}\left(\frac{1}{6}\right)^2 = \frac{(-5)(-1)\,3}{3!}\left(\frac{1}{6}\right)^3 + \frac{(-5)(-1)\,3\cdot 7}{4!}\left(\frac{1}{6}\right)^4 + \cdots$$

$$\frac{55}{72} + 1 + \frac{(-5)}{1!}\left(\frac{1}{6}\right) + \frac{(-5)(-1)}{2!}\left(\frac{1}{6}\right)^2 = 1 + \frac{(-5)}{1!}\left(\frac{1}{6}\right) + \frac{(-5)(-1)}{2!}\left(\frac{1}{6}\right)^2 + \frac{(-5)(-1)\,3}{3!}\left(\frac{1}{6}\right)^3 + \cdots$$

$$\frac{55}{72} + \left(1 - \frac{5}{6} + \frac{5}{72}\right) = (1-x)^{-p/q}$$

$$p = -5,\ q = 4,\ \frac{x}{q} = \frac{1}{6} \Rightarrow x = \frac{2}{3}$$

$$\therefore \frac{55}{72} + \frac{17}{72} = \left[1 - \frac{2}{3}\right]^{5/4}$$

$$\frac{55}{72} = \left(\frac{1}{3}\right)^{5/4} - \frac{17}{72}$$

$$S = \frac{72}{5}\left(\frac{3^{-5/4}(72) - 17}{72}\right)$$

$$= \frac{72}{5}\left[\frac{3^{-5/4}(3^2)\,8 - 17}{72}\right]$$

$$= \frac{72}{5}\left[\frac{3^{3/4}(8) - 17}{72}\right]$$

$$= \frac{72}{5}\left(\frac{8(27)^{1/4} - 17}{72}\right)$$

$$\therefore S = \frac{1}{5}\left[8(27)^{1/4} - 17\right]$$

EXPONENTIAL SERIES

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$

$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} \dots$

adding ① and ② we have

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

subtracting ② from ① we have

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \dots$$

Putting $x=1$ in (2) (3) and (4) we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$\frac{e + e^{-1}}{2} = \frac{1 + \cancel{\frac{1}{1!}} + \frac{1}{2!} + \cancel{\frac{1}{3!}} + 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}}{2}$

$= \frac{2\left(1 + \frac{1}{2!} + \frac{1}{4!}\right)}{2}$

$= 1 + \frac{1}{2!} + \frac{1}{4!}$

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \dots$$

$$\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \dots$$

Problem : 1 Find the coefficient of x^n in $\dfrac{a+be^x+ce^{2x}}{e^{3x}}$

Solution:

$$\frac{a+be^x+ce^{2x}}{e^{3x}} = (a+be^x+ce^{2x})e^{-3x}$$

$$= ae^{-3x}+be^{-2x}+ce^{-x}$$

$$= a\left[1-\frac{3x}{1!}+\frac{(3x)^2}{2!}+\cdots+\frac{(-1)^n(3x)^n}{n!}+\cdots\right]$$

$$+b\left[1-\frac{2x}{1!}+\frac{(2x)^2}{2!}+\cdots+\frac{(-1)^n(2x)^n}{n!}+\cdots\right]$$

$$+c\left[1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{(-1)^n x^n}{n!}+\cdots\right]$$

$\therefore$ coefficient of x^n in $\dfrac{a+be^x+ce^{2x}}{e^{3x}}$ is

$$= \frac{(-1)^n}{n!}\left[a3^n+b2^n+c\right]$$

Problem : 2 What is the coefficient of x^n in the expansion of $(1+x)e^{1+x}$ in ascending powers of x.

Solution:

$$(1+x)e^{1+x} = (1+x)e.e^x$$

$$= e(1+x)\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}+\cdots\right]$$

coefficient of x^n in $(1+x)e^{1+x}$ is $= e\left(\dfrac{1}{n!}+\dfrac{1}{(n-1)!}\right)$

$$= e\left[\frac{1}{n!}+\frac{n}{n!}\right] = \frac{e}{n!}(1+n)$$

Problem : 3 Show that the coefficient of x^n in the expansion of e^{e^x} is $\frac{1}{n!}\left[\frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\cdots\right]$. Hence show that

i). $\frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\cdots = 5e.$

ii) $\frac{1^4}{1!}+\frac{2^4}{2!}+\frac{3^4}{3!}+\cdots = 15e$

Solution :

$$e^{e^x} = e^{(e^x)} = 1+\frac{e^x}{1!}+\frac{e^{2x}}{2!}+\frac{e^{3x}}{3!}+\cdots+\frac{e^{nx}}{n!}+\cdots$$

$\therefore$ coefficient of x^n in e^{e^x} is $= \frac{1}{n!}+\frac{1}{2!}\left(\frac{2^n}{n!}\right)+\frac{1}{3!}\left(\frac{3^n}{n!}\right)+\cdots$

$$= \frac{1}{n!}\left[\frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\cdots\right] \longrightarrow (1)$$

i). From (1) coefficient of x^3 in e^{e^x} is

$$\frac{1}{3!}\left[\frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\cdots\right] \longrightarrow (2)$$

We now find the coefficient of x^3 in e^{e^x} in another way as follows.

$$e^{e^x} = e^{1+x+\frac{x^2}{2}+\cdots}$$

$= e\left(\frac{2}{6}+\frac{1}{2}\right) = e\left(\frac{4+6}{12}\right)$

$= \left(\frac{10}{12}\right)e$

$= e\left(\frac{5}{6}\right)$

$$= e.e^{\left(x+\frac{x^2}{2!}+\cdots\right)}$$

$$= e\left[1+\left[x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right]+\frac{1}{2!}\left(x+\frac{x^2}{2!}+\cdots\right)^2 + \frac{1}{3!}\left[x+\frac{x^2}{2!}+\cdots\right]^3+\cdots\right] \longrightarrow (3)$$

coefficient of x^3 in e^{e^x} from (3)

$$= e\left[\frac{1}{3!}+\frac{1}{2!}+\frac{1}{3!}\right] \longrightarrow (4)$$

$$= \frac{5e}{6}$$

From (2) and (4) we get.

$$\frac{1}{3!}\left[\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!}\right] = \frac{5e}{6}$$

$$\therefore \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} = 5e$$

ii). Similarly by finding the coefficient of x^4 in e^{e^x} in two different ways we can prove

$$\frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + \dots = 15e$$

Problem : 4 prove $\log 2 - \frac{(\log 2)^2}{2!} + \frac{(\log 2)^3}{3!} - \dots = \frac{1}{2}$.

Solution : Put $\log 2 = y$

$$LHS = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \dots$$

$$= -\left[-\frac{y}{1} + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\right]$$

$$e^{-y} = 1 - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots$$

$$= -(e^{-y} - 1)$$

$$= 1 - e^{-\log 2} = 1 - e^{\log 1/2}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

Problem : 5

Prove $$\frac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots}{1 + \frac{1}{3!} + \frac{1}{5!} + \dots} = \frac{e-1}{e+1}$$

Solution :

$$LHS = \frac{\frac{1}{2}(e+e^{-1})-1}{\frac{1}{2}(e-e^{-1})} = \frac{\frac{1}{2}(e+e^{-1}-2e)}{\frac{1}{2}(e-e^{-1})}$$

$$= \frac{e^2+1-2e}{e^2-1} \qquad = \frac{e^2+1-2e}{e^2-1}$$

$$= \frac{(e-1)^2}{(e+1)(e-1)} = \frac{e-1}{e+1} \qquad = \frac{(e-1)^2}{(e+1)(e-1)}$$

Problem : 6 Show that if $a > 1.5$

$$S = 1 + \frac{1+a}{2!} + \frac{1+a+a^2}{3!} + \cdots = \frac{e^a - e}{a-1} \qquad T_n = \frac{a^n-1}{n!(a-1)}$$

Solution : nth term

$$T_n = \frac{1+a+a^2+\cdots+a^{n-1}}{n!} = \frac{a^{n-1}}{n!}\left(\frac{a-1}{a-1}\right) = \frac{a^{n-1+1}-1}{n!(a-1)}$$

$$= \frac{a^n-1}{n!\,(a-1)}$$

$$\therefore T_n = \left(\frac{1}{a-1}\right)\left[\frac{a^n}{n!} - \frac{1}{n!}\right]$$

putting $n = 1, 2, 3 \ldots\ldots$ in (1) we get

$$T_1 = \left(\frac{1}{a-1}\right)\left(\frac{a}{1!} - \frac{1}{1!}\right)$$

$$T_2 = \left(\frac{1}{a-1}\right)\left(\frac{a^2}{2!} - \frac{1}{2!}\right)$$

$$T_3 = \left(\frac{1}{a-1}\right)\left(\frac{a^3}{3!} - \frac{1}{3!}\right)$$

$$\therefore S = \left(\frac{1}{a-1}\right)\left[\left(\frac{a}{1!} + \frac{a^2}{2!} + \cdots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)\right]$$

$$= \frac{1}{(a-1)}\left[(e^a-1) - (e-1)\right]$$

$$= \frac{e^a - e}{a-1}$$

Problem : 7 Prove that

$$S = 1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \cdots = \frac{3e}{2}$$

Solution :

$$n^{th} \text{ term } T_n = \frac{1+2+\cdots+n}{n!}$$

$$= \frac{n(n+1)}{2n!} = \frac{\cancel{n}(n+1)}{2\cancel{n}(n-1)!}$$

$$= \frac{n+1}{2(n-1)!}$$

Let $n+1 = A + B(n-1)$

Putting $n=1 \Rightarrow 1+1 = A \Rightarrow A = 2$

$n = 0 \Rightarrow 1 = 2 + B(-1)$

$1 - 2 = -B$

$B = 1$

$$\therefore T_n = \frac{2+(n-1)}{2(n-1)!}$$

$$T_n = \frac{1}{(n-1)!} + \frac{1}{2(n-2)!}$$

$$\frac{1}{2(1-2)!}$$

Putting $n = 1, 2, 3 \ldots$ in (1) we get

$T_1 = 1 - \frac{1}{2}$

$T_2 = \frac{1}{1!} + \frac{1}{2}$

$T_3 = \frac{1}{2!} + \frac{1}{2}\left(\frac{1}{1!}\right)$

$T_4 = \frac{1}{3!} + \frac{1}{2}\left(\frac{1}{2!}\right)$

$$\therefore S = \left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \frac{1}{2}\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] = e + \frac{1}{2}e =$$

Problem: 8 Prove that

$$S = \sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!} = \frac{e}{2} + \frac{2}{e}$$

Solution: Let $5n+1 = A + B(2n+1)$

Put $n = -1/2 \Rightarrow \frac{-5}{2} + 1 = A \Rightarrow A = -3/2$

$n = 0 \Rightarrow 1 = \frac{-3}{2} + B \Rightarrow B = 1 + \frac{3}{2} = 5/2$

$$\therefore T_{n+1} = \frac{5n+1}{(2n+1)!} = \frac{A + B(2n+1)}{(2n+1)!} = \frac{A}{(2n+1)!} + \frac{B(2n+1)}{(2n+1)!} = \frac{A}{(2n+1)!} + \frac{B}{2n!}$$

$$= \frac{-3}{2}\left(\frac{1}{2n+1!}\right) + \frac{5}{2}\left(\frac{1}{2n!}\right)$$

Putting $n = 0, 1, 2, \ldots$ in (1) we get

$$T_0 = \left(\frac{-3}{2}\right)\left(\frac{1}{1!}\right) + \frac{5}{2}(1)$$

$$T_1 = \left(\frac{-3}{2}\right)\frac{1}{3!} + \left(\frac{5}{2}\right)\frac{1}{2!}$$

$$T_2 = \left(\frac{-3}{2}\right)\frac{1}{5!} + \left(\frac{5}{2}\right)\frac{1}{4!}$$

$$\therefore S = \frac{-3}{2}\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right] + \frac{5}{2}\left[1 + \frac{1}{2!} + \frac{1}{4!} + \cdots\right]$$

$$= \frac{-3}{2}\left[\frac{e - e^{-1}}{2}\right] + \frac{5}{2}\left[\frac{e + e^{-1}}{2}\right]$$

$$= e\left[\frac{5}{4} - \frac{3}{4}\right] + e^{-1}\left[\frac{5}{4} + \frac{3}{4}\right]$$

$$= \frac{e}{2} + \frac{2}{e}$$

Problem: 9 Find $S = \sum_{n=1}^{\infty} \frac{n-1}{(n+2)\,n!} x^n$

Solution:

$$T_n = \frac{n-1}{(n+2)\,n!} x^n$$

$$T_n = \frac{n^2-1}{(n+2)!}x^n$$

Let $n^2-1 = A + B(n+2) + C(n+2)(n+1)$

$n=-2 \Rightarrow 4-1 = A + B(0) + C(0) \Rightarrow A = 3$

$n=-1 \Rightarrow 0 = 3 + B(1) + 0 \Rightarrow B = -3$

$n=0 \Rightarrow -1 = 3 - 3(2) + C(2)(1)$

$-1 = 3 - 6 + 2C$

$-1 = -3 + 2C \qquad 2C = -1 + 3 \Rightarrow C = 1$

$$\therefore T_n = \frac{3}{(n+2)!}x^n - \frac{3}{(n+1)!}x^n + \frac{1}{n!}x^n \longrightarrow (1)$$

Putting $n = 1, 2, \ldots$ in (1) we get

$$T_1 = \frac{3}{3!}x - \frac{3}{2!}x + \frac{1}{1!}x$$

$$T_2 = \frac{3}{4!}x^2 - \frac{3}{3!}x^2 + \frac{1}{2!}x^2$$

$$T_3 = \frac{3}{5!}x^3 - \frac{3}{4!}x^3 + \frac{1}{3!}x^3$$

$$\therefore S = 3\left[\frac{x}{3!} + \frac{x^2}{4!} + \cdots\right] - 3\left[\frac{x}{2!} + \frac{x^2}{3!} + \cdots\right] + \left[\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$$

$$= \frac{3}{x^2}\left[\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right] - \frac{3}{x}\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + \left[\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$$

$$= \frac{3}{x^2}\left(e^x - \frac{x^2}{2!} - \frac{x}{1!} - 1\right) - \frac{3}{x}\left(e^x - \frac{x}{1!} - 1\right) + (e^x - 1)$$

$$= \frac{3}{x^2}e^x - \frac{3}{2} - \frac{3}{x} - \frac{3}{x^2} - \frac{3e^x}{x} + 3 + \frac{3}{x} + e^x - 1$$

$$= e^x\left(\frac{3}{x^2} - \frac{3}{x} + 1\right) - \frac{3}{x^2} + \frac{1}{2}$$

$$= e^x\left(\frac{3-3x+x^2}{x^2}\right) + \frac{x^2-6}{2x^2}$$

$$= \frac{2e^x(x^2-3x+3)+(x^2-6)}{2x^2}$$

Problem : 10 Show that $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \cdots = 27e.$

solution:

n^{th} term $T_n = \frac{n^2(n+1)^2}{n!}$

$$= \frac{n(n+1)^2}{(n-1)!}$$

Let $n(n+1)^2 = A + B(n-1) + C(n-1)(n-2) + D(n-1)(n-2)(n-3).$

$n=1 \Rightarrow 1(2)^2 = A + 0 + 0 + 0 \Rightarrow A = 4$

$n=2 \Rightarrow 2(3)^2 = 4 + B(1) + 0 + 0 \Rightarrow 18 - 4 = B \Rightarrow B = 14$

$n=3 \Rightarrow 3(4)^2 = 4 + 14(2) + C(3-1)(3-2) + 0$

$48 = 4 + 28 + C(2)(1)$

$48 - 32 = 2C$

$16 = 2C \Rightarrow C = 8$

$n = 0$ $\quad 0 = 4 + 14(-1) + 8(-1)(-2) + D(-1)(-2)(-3)$

$0 = 4 - 14 + 16 - 6D$

$6D = 6 \Rightarrow D = 1$

$\therefore T_n = \frac{4}{(n-1)!} + \frac{14}{(n-2)!} + \frac{8}{(n-3)!} + \frac{1}{(n-4)!}$

$T_1 = \frac{4}{1}$

$T_2 = \frac{4}{1!} + 14$

$T_3 = \frac{4}{2!} + \frac{14}{1!} + 8$

$T_4 = \frac{4}{3!} + \frac{14}{2!} + \frac{8}{1!} + 1$

$T_5 = \frac{4}{4!} + \frac{14}{3!} + \frac{8}{2!} + \frac{1}{1!}$

$$\therefore S = 4\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + 14\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + 8\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$= 4e + 14e + 8e + e = 27e$$

Logarithmic series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$\log(1+x) - \log(1-x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^2}{2} + \frac{x^3}{3}$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

$$\log(1+x) + \log(1-x) = -2\left[\frac{x^2}{2} + \frac{x^4}{4} + \cdots\right]$$

Problem.1 Show that

$$\left[\frac{a-b}{a}\right] + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 + \cdots = \log_e a - \log_e b$$

Solution: Put $\frac{a-b}{a} = x$

$$LHS = x + \frac{1}{2}x^2 + \frac{1}{3}x^3$$

$= -\log(1-x)$

$= -\log\left[1-\frac{a-b}{a}\right] = -\log\left(\frac{a-a+b}{a}\right)$

$= -\log\left(\frac{b}{a}\right)$

$= \log\left(\frac{a}{b}\right)$

$= \log a - \log b = RHS$

Problem: 2 Prove that

$$\log\sqrt{\frac{n+1}{n}} = \left[\frac{1}{2n+1}\right] + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots\cdots$$

Solution: Let $\frac{1}{2n+1} = x$ $\quad RHS = x + \frac{x^3}{3} + \frac{x^5}{5}$

$$RHS = \frac{1}{2}\log\left[\frac{1+x}{1-x}\right]$$

$$= \frac{1}{2}\log\left[\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right] = \frac{1}{2}\log\left(\frac{\frac{2n+2}{2n+1}}{\frac{2n}{2n+1}}\right)$$

$$= \frac{1}{2}\log\left(\frac{2n+2}{2n}\right) = \frac{1}{2}\log\left[\frac{2n}{2n}+\frac{2}{2n}\right]$$

$$= \frac{1}{2}\log\left[\frac{n+1}{n}\right]$$

$$= \log\sqrt{\frac{n+1}{n}} = LHS.$$

Problem. 3. Show that

$$\frac{3}{10}\left[\log 10 + \frac{1}{2^7} + \frac{1}{2}\cdot\frac{3}{2^{14}} + \frac{1}{3}\cdot\frac{3^2}{2^{21}} + \cdots\right] = \log 2$$

Solution:

$$LHS = \frac{1}{10}\left[3\log 10 + \left(\frac{3}{2^7}\right) + \frac{1}{2}\left(\frac{3}{2^7}\right)^2 + \frac{1}{3}\left(\frac{3}{2^7}\right)^3 + \cdots\right]$$

$$= \frac{1}{10}\left[\log 10^3 - \log\left(1 - \frac{3}{2^7}\right)\right]$$

$$= \frac{1}{10}\left[\log 1000 - \log\left(\frac{125}{2^7}\right)\right]$$

$$= \frac{1}{10}\log\left[\frac{1000}{125} \times 2^7\right] = \frac{1}{10}\log(2^3 \times 2^7)$$

$$= \frac{1}{10} \times \log 2^{10} = \log 2 = RHS$$

Problem 4. Sum to infinity the series

$$\left(1+\frac{1}{2}\right) + \left(\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{5}+\frac{1}{6}\right)\left(\frac{1}{9^2}\right) + \cdots$$

Solution:

$$\left(1+\frac{1}{2}\right) + \left(\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{5}+\frac{1}{6}\right)\left(\frac{1}{9^2}\right) + \cdots$$

$$= \left[1 + \left(\frac{1}{3}\right)\left(\frac{1}{9}\right) + \frac{1}{5}\left(\frac{1}{9^2}\right) + \cdots\right] + \left[\frac{1}{2} + \frac{1}{4}\left(\frac{1}{9}\right) + \frac{1}{6}\left(\frac{1}{9^2}\right) + \cdots\right]$$

$$= 3\left[\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \cdots\right] + \frac{9}{2}\left[\frac{1}{9} + \frac{1}{2}\left(\frac{1}{9}\right)^2 + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \cdots\right]$$

$$= 3\left[\frac{1}{2}\log\left[\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right]\right] - \frac{9}{2}\log\left(1 - \frac{1}{9}\right)$$

$$= \frac{3}{2}\log 2 - \frac{9}{2}\log\left(\frac{8}{9}\right) = \frac{3}{2}\left[\log 2 - 3\log\left(\frac{8}{9}\right)\right]$$

$$= \frac{3}{2}\left[\log 2 - 3\log 8 + 3\log 9\right]$$

$$= \frac{3}{2}\left[\log 2 - 9\log 2 + 6\log 3\right]$$

$$= \frac{3}{2}\left[6\log 3 - 8\log 2\right] = 9\log 3 - 12\log 2.$$

Problem : 5 prove that $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \cdots$

Solution : put $x = \frac{1}{n+1}$

$$LHS = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

$$= -\log(1-x) = -\log\left[1 - \frac{1}{n+1}\right]$$

$$= -\log\left(\frac{n}{n+1}\right)$$

$$= \log\left(\frac{n+1}{n}\right) = \log\left[1 + \frac{1}{n}\right]$$

$$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots = RHS$$

Problem : 6 If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ prove that

$$x = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

Solution :

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{(ie) } y = \log(1+x)$$

$$\therefore \quad e^y = 1 + x$$

$$x = e^y - 1$$

$$= \left[1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right] - 1$$

$$x = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

Problem : 7 If $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \cdots$ and $|x| < 1$ show that $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Solution :

$$x = y - \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

$$= -\left[1 - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\right]$$

$$= -\left(e^{-y} - 1\right)$$

Thus $x = 1 - e^{-y}$

$$\therefore e^{-y} = 1 - x$$

$$-y = \log_e (1-x)$$

$$y = -\log_e (1-x)$$

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Problem : 8 If $\log(1-x+x^2)$ be expanded in ascending powers of x in the form $a_1x + a_2x^2 + a_3x^3 + \dots$ prove that $a_3 + a_6 + a_9 + \dots = \frac{2}{3}\log 2$.

Solution :

$$\log(1-x+x^2) = \log\left[\frac{1+x^3}{1+x}\right]$$

$$= \log(1+x^3) - \log(1+x)$$

$$= \left[x^3 - \frac{(x^3)^2}{2} + \dots + \frac{(-1)^{n-1}(x^3)^n}{n} + \dots\right]$$

$$- \left[x - \frac{x^2}{2} + \dots + \frac{(-1)^{n-1}x^n}{n} + \dots\right]$$

coefficient of x^{3n} is $a_{3n} = \frac{(-1)^{n-1}}{n} - \frac{(-1)^{3n-1}}{3n}$

$$= \frac{(-1)^{n-1}}{n}\left[1 - \frac{1}{3}\right]$$

$$= (-1)^{n-1}\left[\frac{2}{3n}\right]$$

Putting $n = 1, 2, 3 \ldots\ldots$ in (1) and adding we get

$$a_3 + a_6 + a_9 + \cdots = \frac{2}{3}\left[1 - \frac{1}{2} + \frac{1}{3} - \cdots\right] = \frac{2}{3}\log_e(1+1)$$

$$= \frac{2}{3}\log_e 2.$$

Problem : 9 Show that if $x > 0$

$$\log x = \frac{x-1}{x+1} + \frac{1}{2}\,\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\,\frac{x^3-1}{(x+1)^3} + \cdots$$

Solution :

$$RHS = \left(\frac{x}{x+1}\right) + \frac{1}{2}\left(\frac{x}{x+1}\right)^2 + \frac{1}{3}\left(\frac{x}{x+1}\right)^3 + \cdots$$

$$+ \left[-\left(\frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)^2 - \frac{1}{3}\left(\frac{1}{x+1}\right)^3 - \cdots\cdots\right]$$

$$= -\log\left[1 - \frac{x}{x+1}\right] + \log\left[1 - \frac{1}{x+1}\right]$$

$$= -\log\left(\frac{1}{x+1}\right) + \log\left(\frac{x}{x+1}\right) = -\log 1 + \log(x+1) + \log x - \log(x+1)$$

$$= 0 + \log x$$

$$= \log x = LHS$$

Problem : 10. If a, b, c denote three consecutive integers prove that $\log b = \frac{1}{2}\log a + \frac{1}{2}\log c + \left[\frac{1}{2ac+1}\right] + \frac{1}{3}\left[\frac{1}{2ac+1}\right]^3 + \cdots\cdots$

Solution :

$$RHS = \frac{1}{2}\log a + \frac{1}{2}\log c + \frac{1}{2}\log\left[\frac{1 + \frac{1}{2ac+1}}{1 - \frac{1}{2ac+1}}\right] \qquad \frac{2ac+1+1}{2ac+1-1}$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\left(\frac{ac+1}{ac}\right)$$

$$= \frac{1}{2}\log(ac+1)$$

$$= \frac{1}{2}\log[(b-1)(b+1)+1]$$

(since a, b, c are consecutive integers $a = b-1$ and $c = b+1$)

$$= \frac{1}{2}\log b^2$$

$$= \log b = LHS$$

Problem : 11 If $F(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$ where $-1 < x < 1$

i). represent $F(x)$ as a logarithmic function

ii). Hence prove $F\left(\frac{2x}{1+x^2}\right) = 2F(x)$

solution:

i). For $-1 < x < 1$ we have

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\therefore \log(1+x) - \log(1-x) = 2\left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots\right]$$

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$$

$$\therefore F(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

ii. Now, $F\left(\frac{2x}{1+x^2}\right) = \frac{1}{2}\log\left(\dfrac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$

$$= \frac{1}{2}\log\left(\frac{1+x^2+2x}{1+x^2-2x}\right) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)^2$$

$$= 2F(x)$$

Problem : 12 Sum the series to infinity

$\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \cdots$

Solution:

$\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \cdots$

$$= \frac{1}{\log_e 3} - \frac{1}{\log_e 9} + \frac{1}{\log_e 27} - \frac{1}{\log_e 81} + \cdots$$

$$= \frac{1}{\log_e 3} - \frac{1}{2\log_e 3} + \frac{1}{3\log_e 3} - \frac{1}{4\log_e 3} + \cdots$$

$$= \frac{1}{\log 3}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right] \quad \log(1+1)$$

$$= \frac{\log_e 2}{\log_e 3} = \log_e 2 \times \log_3 e = \log_3 2.$$

Problem : 13 Show that $(1+x)^{1+x} = 1 + x + x^2 + \frac{1}{2}x^3$ neglecting x^4 and higher powers of x. Also find an approximate value of $(1.01)^{1.01}$

Solution:

$$(1+x)^{1+x} = e^{\log(1+x)^{1+x}}$$

$$= e^{(1+x)\log(1+x)}$$

$$= e^{(1+x)\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)}$$

$$= e^{x + \frac{1}{2}x^2 - \frac{1}{6}x^3}$$

$$= 1 + \left[x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right] + \frac{1}{2!}\left[x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right]^2 + \frac{1}{3!}\left[x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right]^3$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{2!}(x^2 + x^3) + \frac{1}{3!}x^3$$

$$= 1 + x + x^2 + \frac{1}{2}x^3$$

Put $x = 0.01$ in the result

$(1.01)^{1.01} = 1 + .01 + .0001 + \frac{1}{2}(.000001)$

$= 1.0101005.$

Problem : 14. Show that $S = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} = \log 2 - \frac{1}{2}$

Solution :

Here the n^{th} term $T_n = \dfrac{1}{(2n-1)(2n)(2n+1)}$

Let $T_n = \dfrac{A}{2n-1} + \dfrac{B}{2n} + \dfrac{C}{2n+1}$ $\quad 1 = A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n)(2n-1)$

$1 = 0 + 0 + B(-1)(1)$

$A = \frac{1}{2},\ B = -1,\ C = \frac{1}{2}$

$\therefore\ T_n = \left(\frac{1}{2}\right)\left(\frac{1}{2n-1}\right) - \frac{1}{2n} + \frac{1}{2}\left(\frac{1}{2n+1}\right) \longrightarrow (1)$

Putting $n = 1, 2, 3 \ldots$ in (1) we get

$T_1 = \frac{1}{2}\left(\frac{1}{1}\right) - \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3}\right)$

$T_2 = \frac{1}{2}\left(\frac{1}{3}\right) - \frac{1}{4} + \frac{1}{2}\left(\frac{1}{5}\right)$

$T_3 = \frac{1}{2}\left(\frac{1}{5}\right) - \frac{1}{6} + \frac{1}{2}\left(\frac{1}{7}\right)$

$\therefore\ S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$

$= \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right] - \frac{1}{2}$

$= \log 2 - \frac{1}{2}.$

$S = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots\right)$

$- \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)$

$+ \frac{1}{2}\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right)$

$= \frac{1}{2}\left[\frac{1}{2}\log\left(\frac{1+1}{1-1}\right) + \log(1-1) + \frac{1}{2}\log\left(\ldots\right.\right.$

$= \frac{1}{2}\left[\frac{1}{2}\left[\log(2) - \log(0)\right] + \frac{1}{2}\left[\log 2 - 1\right.\right.$

$= \frac{1}{2}\left[\frac{1}{2}(\log 2 - 1) + \frac{1}{2}(\log 2 - 1)\right.$

$= \frac{1}{2} \times 2\left(\frac{1}{2}\log 2 - 1\right) = \log 2 \ldots$

Problem : 15 Prove $S = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 7} + \cdots = 2 - \log 2$

Solution :

Here $T_n = \dfrac{1}{n(2n+1)}$

$$T_n = \frac{A}{n} + \frac{B}{2n+1}$$

we can Find $A=1, B=2$

$$\therefore T_n = \frac{1}{n} - \frac{2}{2n+1}$$

putting $n = 1, 2, 3 \ldots$ in(1) we get

$$T_1 = \frac{1}{1} - \frac{2}{3}$$

$$T_2 = \frac{1}{2} - \frac{2}{5}$$

$$T_3 = \frac{1}{3} - \frac{2}{7}$$

$$\therefore S = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$

$$= 1 - \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$$

$$= 1 - [\log 2 - 1]$$

$$= 2 - \log 2.$$

$= \frac{1}{3}$

$= \frac{4}{10}$

$-\left(-\frac{1}{1}+\frac{1}{2}-\frac{1}{3}\right) - \log(1-x)$

$S = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\right) - 2\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right]$

$= -\log 0 - \log\left(\frac{1+1}{1-1}\right)$

$= -1 - [\log(2) - \log(0)]$

$= -1 - \log 2 + \log 0$

Problem: 16 Show that $S = \dfrac{5}{1\cdot2\cdot3} + \dfrac{7}{3\cdot4\cdot5} + \dfrac{9}{5\cdot6\cdot7} + \cdots = 3\log 2 - 1$

Solution:

Here $T_n = \dfrac{2n+3}{(2n-1)2n(2n+1)}$

$2n+3 = A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n)(2n\ldots$

$3 = B($

$$T_n = \frac{A}{2n-1} + \frac{B}{2n} + \frac{C}{2n+1}$$

we can Find $A = 2,\ B = -3,\ C = 1$

$$T_n = \frac{2}{2n-1} - \frac{3}{2n} + \frac{1}{2n+1}$$

$$= 2\left(\frac{1}{2n-1} - \frac{1}{2n}\right) - \left(\frac{1}{2n} - \frac{1}{2n+1}\right) \longrightarrow (1)$$

putting $n = 1, 2, 3 \ldots$ in (1) we get

$T_1 = 2\left(\frac{1}{1} - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)$

$T_2 = 2\left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{4} - \frac{1}{5}\right)$

$\therefore S = 2 + 3\left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

$= 2 + 3(\log 2 - 1)$

$= 3\log 2 - 1.$

Problem : 17 prove $S = \frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \cdots\cdots = \log 4 - 1$

Solution:

$$T_n = (-1)^{n-1}\left[\frac{1}{n(n+1)}\right]$$

we have $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$T_n = (-1)^{n-1}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

putting $n = 1, 2, 3 \ldots\ldots$ in (1) we get

$T_1 = 1 - \frac{1}{2}$

$T_2 = -\frac{1}{2} + \frac{1}{3}$

$T_3 = \frac{1}{3} - \frac{1}{4}$

$\therefore S = 1 + 2\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right)$

$= 1 + 2(\log 2 - 1)$

$= \log 4 - 1$

Problem : 18 prove that

$$\log\left(1 + \frac{1}{n}\right)^n = 1 - \frac{1}{2(n+1)} - \frac{1}{2\cdot 3(n+1)^2} - \frac{1}{3\cdot 4(n+1)^3} - \cdots\cdots$$

Solution: put $\frac{1}{n+1} = x$

$$RHS = 1 - \frac{1}{2}x - \frac{1}{2\cdot 3}x^2 - \frac{1}{3\cdot 4}x^3 - \cdots$$

$$= 1 - \left(1-\frac{1}{2}\right)x - \left(\frac{1}{2}-\frac{1}{3}\right)x^2 - \left(\frac{1}{3}-\frac{1}{4}\right)x^3 - \cdots$$

$$= \left[-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots\right] + \left[1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots\right]$$

$$= -\left[x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right] + \frac{1}{x}\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$$

$$= \log(1-x) - \frac{1}{x}\log(1-x)$$

$$= \left(1 - \frac{1}{x}\right)\log(1-x)$$

$$= (1-n-1)\log\left(1 - \frac{1}{n+1}\right) \quad \left(\text{since } x = \frac{1}{n+1}\right)$$

$$= -n\log\left(\frac{n}{n+1}\right)$$

$$= \log\left(\frac{n+1}{n}\right)^n = LHS$$

Problem : 19 If $a+b+c=0$ show that

i). $$\frac{a^5+b^5+c^5}{5} = \left[\frac{a^3+b^3+c^3}{3}\right]\left[\frac{a^2+b^2+c^2}{2}\right]$$

ii). $$\frac{a^7+b^7+c^7}{7} = \left[\frac{a^4+b^4+c^4}{2}\right]\left[\frac{a^3+b^3+c^3}{3}\right]$$

Solution: consider the product.

$$(1-ax)(1-bx)(1-cx) = 1-(a+b+c)x + (ab+bc+ca)x^2 - (abc)x$$

$$= 1 + px^2 - qx^3 \quad (\because a+b+c=0)$$

where $p = ab+bc+ca$ and $q = abc$

$\log(1-ax)+\log(1-bx)+\log(1-cx) = \log(1+px^2-qx^3)$

$= \log[1-(qx-p)x^2] \longrightarrow (1)$

If x is sufficiently small (1) can be expanded in series

$ax+\frac{a^2x^2}{2}+\frac{a^3x^3}{3}+\cdots+bx+\frac{b^2x^2}{2}+\frac{b^3x^3}{3}+\cdots+cx+\frac{c^2x^2}{2}+\frac{c^3x^3}{3}+\cdots$

$$= x^2(qx-p)+\frac{x^4(qx-p)^2}{2}+\frac{x^6(qx-p)^3}{3}+\longrightarrow (2)$$

i). comparing the coefficient of x^5, x^3 and x^2 respectively in (2) on both sides we get

$$\frac{a^5+b^5+c^5}{5} = -pq \longrightarrow (3)$$

$$\frac{a^3+b^3+c^3}{3} = q \longrightarrow (4)$$

$$\frac{a^2+b^2+c^2}{2} = -p \longrightarrow (5)$$

From (3), (4) and (5) we get

$$\frac{a^5+b^5+c^5}{5} = \left[\frac{a^3+b^3+c^3}{3}\right]\left[\frac{a^2+b^2+c^2}{2}\right]$$

ii). comparing the coefficients of x^7, x^4 and x^3 respectively in (2) on both sides we get

$$\frac{a^7+b^7+c^7}{7} = qp^2 \longrightarrow (A)$$

$$\frac{a^4+b^4+c^4}{4} = \frac{p^2}{2}$$

$$\frac{a^4+b^4+c^4}{2} = p^2 \longrightarrow (B)$$

$\frac{a^3+b^3+c^3}{3} = q \longrightarrow$ (C)

From (A), (B) and (C) we get

$$\frac{a^7+b^7+c^7}{7} = \left[\frac{a^4+b^4+c^4}{2}\right]\left[\frac{a^3+b^3+c^3}{3}\right]$$

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chapter I UNIT - IV

Applications of Demoivre's Theorem

Expression for $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$

Theorem:

For any positive integer n *

i). $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \cdots$

ii). $\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots$

Proof: By De moivre's theorem $(x+a)^n = x^n + {}^nC_1 x^{n-1} a$

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + {}^nC_1 \cos^{n-1}\theta (i\sin\theta) + {}^nC_2 \cos^{n-2}\theta$$
$$+ {}^nC_3 \cos^{n-3}\theta (i\sin\theta)^3 + $$

$$= \cos^n\theta + i\,{}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_2 \cos^{n-2}\theta$$
$$- i\,{}^nC_3 \cos^{n-3}\theta \sin^3\theta$$

Equating real and imaginary parts we get (i) and (ii) respectively.

Corollary. $\tan n\theta = \dfrac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + \cdots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta + \cdots}$

Proof:

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$\frac{{}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots}{}$$

Unit - II

2. Relation between Roots and coefficients.

Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be the n roots of the n^{th} degree equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0 \quad \rightarrow (1)$$

Then $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$

$$= a_0 (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\ldots\ldots(x - \alpha_{n-1})(x - \alpha_n)$$

$$= a_0 \left[x^n - x^{n-1}(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + \ldots + x^{n-2}(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \ldots) + x^{n-3}(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4 + \ldots) + \ldots + (-1)^n \alpha_1\alpha_2\ldots\alpha_n\right]$$

Equating coefficients of corresponding powers of x on both sides we get

$\alpha^3 + \beta^3 = (\alpha+\beta) - 3\alpha\beta(\alpha+\beta)$

$\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^2 - 2ab = a^2 + b^2$

$(\alpha+\beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$

$$(-1)^1 a_0 \Sigma\alpha_1 = a_1$$

$$(-1)^2 a_0 \Sigma\alpha_1\alpha_2 = a_2$$

$$(-1)^3 a_0 \Sigma\alpha_1\alpha_2\alpha_3 = a_3$$

$$\vdots \qquad \vdots$$

$$(-1)^n a_0 (\alpha_1\alpha_2\ldots\ldots\alpha_n) = a_n$$

Rewriting the above relations we have

$$S_1 = \Sigma \alpha_1 = (-1)^1 \left(\frac{a_1}{a_0}\right)$$

$$S_2 = \Sigma \alpha_1 \alpha_2 = (-1)^2 \left(\frac{a_2}{a_0}\right)$$

$$S_3 = \Sigma \alpha_1 \alpha_2 \alpha_3 = (-1)^3 \left(\frac{a_3}{a_0}\right)$$

$$\vdots$$

$$S_n = \alpha_1 \alpha_2 \ldots \alpha_n = (-1)^n \left(\frac{a_n}{a_0}\right)$$

Where S_i denotes the sum of the products of roots taken i at a time. In particular, if a quadratic equation.

$ax^2 + bx + c = 0$ has roots α and β then

$$S_1 = \alpha + \beta = -\frac{b}{a}$$

$$S_2 = \alpha\beta = \frac{c}{a}$$

If a cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ

$$S_1 = \overset{\Sigma\alpha}{\alpha + \beta + \gamma} = \frac{-b}{a}$$

$$S_2 = \overset{\Sigma\alpha\beta}{\alpha\beta + \beta\gamma + \gamma\alpha} = \frac{c}{a}$$

$$S_3 = \overset{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma} = \frac{-d}{a}$$

Problem 1 :

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $x^n - P_1 x^{n-1} + P_2 x^{n-2} - \ldots + (-1)^n P_n = 0$ Find the value of $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n)$.

Soln:

For the given equation we have

$$\Sigma \alpha_1 = P_1 \quad \frac{-b}{a} = \frac{-(-P_1)}{1} = P_1$$

$$\Sigma \alpha_1 \alpha_2 = P_2 \quad \frac{c}{a} = \frac{P_2}{1} = P_2$$

$$\vdots$$

$$\alpha_1 \alpha_2 \ldots \alpha_n = P_n$$

$$(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n) = 1 + \Sigma\alpha_1 + \Sigma\alpha_1\alpha_2 + \ldots + (\alpha_1\alpha_2 \ldots \alpha_n)$$

$$= 1 + P_1 + P_2 + \ldots + P_n$$

Problem 2: If α, β, γ are the roots of the equation $x^3 + ax + b = 0$ find the value of

(i) $\Sigma\left(\frac{\alpha}{\beta\gamma}\right)$ * (ii) $\Sigma\left(\frac{\alpha\beta}{\gamma}\right)$ (iii) $\Sigma\left(\frac{\alpha}{\beta+\gamma}\right)$

(iv) $\Sigma^* \frac{1}{\beta+\gamma}$ (v) $\Sigma\left(\frac{\beta}{\gamma}+\frac{\gamma}{\beta}\right)$ (vi) $\Sigma\alpha^3$

Soln:

$ax^3 + bx^2 + cx + d = 0$

$$\because \Sigma\alpha = 0 \longrightarrow ① \quad \frac{-b}{a} = \frac{0}{1} = 0$$

$$\Sigma\alpha\beta = a \longrightarrow ② \quad c/a = \frac{a}{1}$$

$$\Sigma\alpha\beta\gamma = -b \longrightarrow ③ \quad \frac{-d}{a} = \frac{-b}{1}$$

(i) $\Sigma\left(\frac{\alpha}{\beta\gamma}\right) = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta}$

$$= \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$$

$$= \frac{(\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)}{\alpha\beta\gamma}$$

$$\Sigma\left(\frac{\alpha}{\beta\gamma}\right) = \frac{0-2a}{-b} = \frac{+2a}{b}.$$

(ii) $\Sigma\left(\frac{\alpha\beta}{\gamma}\right) = \frac{\alpha\beta}{\gamma} + \frac{\gamma\alpha}{\beta} + \frac{\beta\gamma}{\alpha}$

$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$(\Sigma\alpha)^2 = (\Sigma\alpha)^2 - 2\Sigma(\beta\alpha)$

$$= \frac{(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2}{\alpha\beta\gamma}$$

$$= \frac{\Sigma\alpha^2\beta^2}{\alpha\beta\gamma}$$

? $$= \frac{(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)}{\alpha\beta^2\gamma} = \frac{a^2}{b}$$

(iii) $\Sigma\left(\frac{\alpha}{\beta+\gamma}\right) = \frac{\alpha}{\beta+\gamma} + \frac{\beta}{\gamma+\alpha} + \frac{\gamma}{\alpha+\beta}$

$\alpha+\beta+\gamma=0$

$\alpha+\beta=-\gamma$ $$= \frac{\alpha}{-\alpha} + \frac{\beta}{-\beta} + \frac{\gamma}{-\gamma} = -1-1-1$$

$\beta+\gamma=-\alpha$

$\gamma+\alpha=-\beta$ $$= -3.$$

(iv) $\Sigma\left(\frac{1}{\beta+\gamma}\right) = \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha} + \frac{1}{\alpha+\beta}$

$= \frac{1}{-\alpha} + \frac{1}{-\beta} + \frac{1}{-\gamma}$

$= \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$

$= -\frac{a}{b}$

(v) $\Sigma\left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right) = \Sigma\left(\frac{\beta^2+\gamma^2}{\beta\gamma}\right)$

$= \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{\alpha(\beta^2+\gamma^2)+\beta(\gamma^2+\alpha^2)+\gamma(\alpha^2+\beta^2)}{\alpha\beta\gamma}$

$= \frac{\alpha\beta^2+\alpha\gamma^2+\beta\gamma^2+\beta\alpha^2+\gamma\alpha^2+\gamma\beta^2}{\alpha\beta\gamma}$

$= \frac{\Sigma\alpha^2\beta}{\alpha\beta\gamma} \longrightarrow (A)$

mutiplying equation (1) and (2) we get

$\overset{\Sigma\alpha}{(\alpha+\beta+\gamma)}\overset{\Sigma\alpha\beta}{(\alpha\beta+\beta\gamma+\gamma\alpha)} = 0$

$\alpha^2\beta+\alpha\beta\gamma+\alpha^2\gamma+\alpha\beta^2+\beta^2\gamma+\alpha\beta\gamma+\alpha\beta\gamma+\beta\gamma^2+\alpha\gamma^2 = 0$

$\Sigma\alpha^2\beta + 3\alpha\beta\gamma = 0$

$\Sigma\alpha^2\beta = -3\alpha\beta\gamma.$

From (B) we get

$$\Sigma\left(\frac{\beta}{\gamma}+\frac{\gamma}{\alpha}\right) = \frac{-3\alpha\beta\gamma}{\alpha\beta\gamma} = -3.$$

(vi) $\Sigma\alpha^3 = \alpha^3+\beta^3+\gamma^3$

$$= (\Sigma\alpha)^3 - 3(\Sigma\alpha\beta)(\Sigma\alpha)$$

$$= 0$$

---

problem 2:

If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+px^3+qx^2+rx+s$ find

(i) $\Sigma\left(\frac{1}{\alpha}\right)$ (ii) $\Sigma\left(\frac{\alpha}{\beta}\right)$ (iii) $\Sigma\left(\frac{1}{\alpha\beta}\right)$ (iv) $\Sigma\alpha^2$

(v) $\Sigma\alpha^2\beta$ (vi) $\Sigma\alpha^2\beta\gamma$ (vii) $\Sigma\alpha^2\beta^2$ (viii) $\Sigma\alpha^3$

(ix) $\Sigma\alpha^4$ (x) $\Sigma\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$

(xi) $(\alpha+\beta+\gamma)(\beta+\gamma+\delta)(\gamma+\delta+\alpha)(\delta+\alpha+\beta)$

<u>Soln:</u>

$$\Sigma\alpha = -p \longrightarrow (1)$$

$$\Sigma\alpha\beta = q \longrightarrow (2)$$

$$\Sigma\alpha\beta\gamma = -r \longrightarrow (3)$$

$$\Sigma\alpha\beta\gamma\delta = s \longrightarrow (4)$$

(i) $\Sigma\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

$$= \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}$$

$$= \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{-r}{s}$$

(ii) To find $\Sigma\left(\frac{\alpha}{\beta}\right)$

consider $\Sigma\alpha\,\Sigma\left(\frac{1}{\alpha}\right) = -p\left(\frac{-r}{s}\right)$

$$(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right) = \frac{pr}{s}$$

$$4 + \Sigma\left(\frac{\alpha}{\beta}\right) = \frac{pr}{s}$$

$$\Sigma\left(\frac{\alpha}{\beta}\right) = \frac{pr}{s} - 4$$

(iii) $\Sigma\left(\frac{1}{\alpha\beta}\right) = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\delta} + \frac{1}{\gamma\delta} + \frac{1}{\beta\delta}$

$$= \frac{\gamma\delta + \alpha\delta + \beta\delta + \beta\gamma + \beta\alpha + \alpha\gamma}{\alpha\beta\gamma\delta}$$

$$= \frac{\Sigma\alpha\beta}{\alpha\beta\gamma\delta} = \frac{q}{s}$$

(vi) $\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$$= (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$$

$$= p^2 - 2q$$

(v) To find $\Sigma\alpha^2\beta$. From ① & ② we have

$$(\Sigma\alpha)(\Sigma\alpha\beta) = -pq$$

$$(\alpha+\beta+\gamma+\delta)(\alpha\beta+\alpha\gamma+\gamma\delta+\beta\gamma+\beta\delta+\gamma\delta) = -pq$$

$$\Sigma\alpha^2\beta + 3\Sigma\alpha\beta\gamma = -pq$$

$$\Sigma\alpha^2\beta = -pq - 3\Sigma\alpha\beta\gamma$$

$$= -pq + 3r$$

(vi) To find $\Sigma\alpha^2\beta\gamma$ from ① & ③ we have

$$(\Sigma\alpha)(\Sigma\alpha\beta\gamma) = pr.$$

$$(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta) = pr.$$

$$\Sigma\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta = pr.$$

$$\Sigma\alpha^2\beta\gamma = pr - 4\alpha\beta\gamma\delta$$

$$= pr - 4s$$

(vii) To find $\Sigma\alpha^2\beta^2$. From ② we have $(\Sigma\alpha\beta)^2 = q^2$

$$\Sigma\alpha^2\beta^2 = q^2 - 2\Sigma\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$$

$$= q^2 - 2(pr - 4s) - 6s$$

$$= q^2 - 2pr + 2s$$

(viii) To find $\Sigma\alpha^3$.

From ① and (iv) we have

$$(\Sigma\alpha)(\Sigma\alpha^2) = -p(p^2-2q)$$

$$\Sigma\alpha^3 + \Sigma\alpha^2\beta = 2pq - p^3$$

$$\Sigma\alpha^3 = 2pq - p^3 - (\Sigma\alpha^2\beta)$$

$$= 2pq - p^3 - (3r - pq)$$

$$= 3pq - p^3 - 3r$$

(ix) To find $\Sigma\alpha^4$.

From (iv) we have $(\Sigma\alpha^2)^2 = (p^2-2q)^2$

$$\Sigma\alpha^4 + 2\Sigma\alpha^2\beta^2 = (p^2-2q)^2$$

$$\Sigma\alpha^4 = (p^2-2q)^2 - 2[\Sigma\alpha^2\beta^2]$$

$$= (p^2-2q)^2 - 2(q^2 - 2pr + 2s).$$

$$= p^4 + 4q^2 - 4p^2q + 4pr - 2q^2 - 4s$$

$$= p^4 - 4p^2q + 2q^2 + 4pr - 4s$$

(x) $\Sigma\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = \Sigma\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)$

$$= \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\alpha^2+\gamma^2}{\alpha\gamma} + \frac{\alpha^2+\delta^2}{\alpha\delta} + \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\beta^2+\delta^2}{\beta\delta} + \frac{\gamma^2+\delta^2}{\gamma\delta}$$

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} + \frac{\alpha}{\delta} + \frac{\delta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\beta}{\delta} + \frac{\delta}{\beta} + \frac{\gamma}{\delta} + \frac{\delta}{\gamma}$$

$$= \Sigma\left(\frac{\alpha}{\beta}\right)$$

$$= \frac{pr}{s} - 4$$

(xi) $(\alpha+\beta+\gamma)(\beta+\gamma+\delta)(\gamma+\delta+\alpha)(\delta+\alpha+\beta)$

$$= (-p-\delta)(-p-\alpha)(-p-\beta)(-p-\gamma)$$

$$= (p+\delta)(p+\alpha)(p+\beta)(p+\gamma)$$

$$= p^4 + (\Sigma\alpha)p^3 + (\Sigma\alpha\beta)p^2 + (\Sigma\alpha\beta\gamma)p + \alpha\beta\gamma\delta$$

$$= p^2q - pr + s.$$

Problem 4 :

If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + ax^2 + bx + c = 0$ the value of $\Sigma\left(\frac{\beta+\gamma+\delta-\alpha}{2\alpha^2}\right)$.

<u>Soln</u>:

We have
$$\left.\begin{aligned} \Sigma\alpha &= 0 \quad \tfrac{-b}{a} \\ \Sigma\alpha\beta &= a \quad \tfrac{c}{a} \\ \Sigma\alpha\beta\gamma &= -b \quad \tfrac{-d}{a} \\ \alpha\beta\gamma\delta &= c \quad \tfrac{e}{a} \end{aligned}\right\} \longrightarrow ①$$

$\alpha - 2\alpha$ $\quad$ 0

$$\Sigma\left(\frac{\beta+\gamma+\delta-\alpha}{2\alpha^2}\right) = \Sigma\left(\frac{\alpha+\beta+\gamma+\delta-2\alpha}{2\alpha^2}\right)$$

$$= \Sigma\left(\frac{0-2\alpha}{2\alpha^2}\right) = -\Sigma\left(\frac{2\alpha}{2\alpha^2}\right)$$

$$= -\Sigma\left(\frac{1}{\alpha}\right)$$

$$= -\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)$$

$$= -\left[\frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}\right]$$

$$= -\frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta}$$

$$= -\left(\frac{-b}{c}\right) = \frac{b}{c}.$$

Problem 5: Show that the equation $x^3+qx+r=0$ will have one root twice another if $343r^2+36q^3=0$.

soln:

Let the roots be $\alpha, 2\alpha, \beta$.

$x^3+0x^2+qx+r=0$

$ax^3-bx^2+cx-d=0$

$S_1 = \alpha+2\alpha+\beta = 0 \quad -\frac{b}{a}$

$\beta = -3\alpha \longrightarrow (1)$

$S_2 = 2\alpha^2+2\alpha\beta+\alpha\beta = q \quad c/a$

$2\alpha^2+3\alpha\beta = q \longrightarrow (2)$

using (1) in (2) $\Rightarrow$ $2\alpha^2+3\alpha(-3\alpha)=q$

$2\alpha^2-9\alpha^2=q$

$-7\alpha^2=q \longrightarrow (3)$

Also $S_3 = 2\alpha^2\beta = -r \quad -d/a$

$2(-\alpha^2)(-3\alpha) = -r$

$6\alpha^3 = r$ (using 1)

$\alpha^3 = \frac{r}{6} \longrightarrow (4)$

$(-7\alpha^2)^3 = q^3$

From (3) $\Rightarrow$ $-7^3\alpha^6 = q^3 \longrightarrow (5)$

$-7^3(\alpha^3)^2 = q^3$

$$\begin{array}{r} 7 \\ 7 \\ \hline 49 \\ 7 \\ \hline 343 \end{array}$$

using (4) in (5) $\Rightarrow$ $-343\left(\frac{r^2}{36}\right)=q^3$

$343r^2+36q^3=0$.

If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$

find $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$.

Soln:

$$x^3 + px^2 + qx + r = 0$$

$$\Sigma\alpha = -p$$

$$\Sigma\alpha\beta = q$$

$$\alpha\beta\gamma = -r$$

$$= (1+\beta^2+\alpha^2+\alpha^2\beta^2)(1+\gamma^2) = 1+\alpha^2+\beta^2+\alpha^2\beta^2+\gamma^2+\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2\gamma^2$$

$$(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = 1 + \Sigma\alpha^2 + \Sigma\alpha^2\beta^2 + \alpha^2\beta^2\gamma^2$$

$$= 1 + \left[(\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)\right] + \left[(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)\right] + (\alpha\beta\gamma)^2$$

$$= 1 + \left[(-p)^2 - 2q\right] + \left[q^2 - 2(-r)(-p)\right] + (-r)^2$$

$$= 1 + p^2 - 2q + q^2 - 2pr + r^2.$$

Aliter :

Since α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 + px^2 + qx + r \longrightarrow (1)$$

putting $x = i$ and $x = -i$ in (1)

$$(i-\alpha)(i-\beta)(i-\gamma) = -i - p + qi + r$$

$$= (r-p) + i(q-1) \longrightarrow (2)$$

$$-(i+\alpha)(i+\beta)(i+\gamma) = (r-p) - i(q-1) \longrightarrow (3)$$

Multiplying ② and ③.

$$(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = (r-p)^2 + (q-1)^2$$

$$= r^2+p^2-2rp+q^2+1-2q$$

Problem 7 :

Show that the roots of the equation $px^3+qx^2+rx+s=0$ are in arithmetic progression if $2q^3+27p^2s=9pqr$.

Hence solve $x^3-12x^2+39x-28=0$.

Soln:

Let the roots of the given equation be $a-d,\ a,\ a+d$.

Sum of the roots $= (a-d)+a+(a+d) = \frac{-q}{p}$

$$3a = \frac{-q}{p}. \Rightarrow a = \frac{-q}{3p} \longrightarrow ①$$

Since a is a root.

$$pa^3+qa^2+ra+s=0. \longrightarrow ②$$

Using ① in ②

$$p\left(\frac{-q}{3p}\right)^3 + q\left(\frac{-q}{3p}\right)^2 + r\left(\frac{-q}{3p}\right) + s = 0$$

$$-\frac{pq^3}{27p^3} + \frac{q^3}{9p^2} - \frac{rq}{3p} + s = 0$$

$$\underline{-pq^3 + 3pq^3 - 9p^2qr + 27p^3s} = 0$$

$$27p^3$$

$$2pq^3 - 9p^2qr + 27p^3s = 0$$

$$\div p \Rightarrow \; 2q^3 - 9pqr + 27p^2s = 0$$

$$2q^3 + 27p^2s = 9pqr$$

This is the required condition.

Conversely, suppose $2q^3 + 27p^2s = 9pqr$.

Retracing the steps we get $a = \frac{-q}{3p}$ is one root of the given equation.

$x + \frac{q}{3p}$ is a factor.

Dividing the given by $x + \frac{q}{3p}$ we get that quotient $q(x)$ as

$$q(x) = px^2 + \frac{2q}{3}x + \left(r - \frac{2q^2}{9p}\right).$$

Let α and β be the roots of $q(x) = 0$.

The sum of the roots $\alpha + \beta = 2\left(\frac{-q}{3p}\right) = 2a$

α, a, β are in the A.P.

Comparing $x^3-12x^2+39x-28=0$ with the given
$px^3+qx^2+rx+s=0$
equation $p=1$ $q=-12$ $r=39$ $s=-28$

$$2q^3+27p^2s-9pqr = 2(-12)^3+27(-28)-9(-12)(39)$$

$$= -3456-756+4212$$

$$= 4212-4212 = 0$$

The roots are in A.P and $a = \frac{-q}{3p} = 4$

$x-4$ is a factor.

By actual division we find $x^2-8x+7=0$ is another factor which can be factored as $(x-7)(x-1)$.

The roots are 1, 4, 7.

Problem 8 :

If α, β, γ are the roots of the equation $x^3+px+q=0$ prove that the roots of the equation $x^3+9px-27q=0$ are $\beta+\gamma-2\alpha$, $\gamma+\alpha-2\beta$, $\alpha+\beta-2\gamma$.

Soln:

α, β, γ are the roots.

$\Sigma\alpha = \alpha+\beta+\gamma = 0 \longrightarrow (1)$

$\Sigma\alpha\beta = \alpha\beta+\beta\gamma+\gamma\alpha = p \longrightarrow (2)$

$\alpha\beta\gamma = -q \longrightarrow (3)$

the required equation

$$S_1 = (\beta+\gamma-2\alpha) + (\gamma+\alpha-2\beta) + (\alpha+\beta-2\gamma) = 0$$

$$S_2 = (\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta) + (\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma)$$
$$+ (\alpha+\beta-2\gamma)(\beta+\gamma-2\alpha)$$

$-2\alpha = \alpha - 3\alpha$

$$= (\alpha+\beta+\gamma-3\alpha)(\alpha+\beta+\gamma-3\beta) +$$
$$(\alpha+\beta+\gamma-3\beta)(\alpha+\beta+\gamma-3\gamma) +$$
$$(\alpha+\beta+\gamma-3\gamma)(\alpha+\beta+\gamma-3\alpha)$$

$$= 9(\alpha\beta+\beta\gamma+\gamma\alpha) \quad \text{(using 1)}$$

$$= 9p$$

$$S_3 = (\beta+\gamma-2\alpha)(\alpha+\gamma-2\beta)(\alpha+\beta-2\gamma)$$

$$= (\alpha+\beta+\gamma-3\alpha)(\alpha+\beta+\gamma-3\beta)(\alpha+\beta+\gamma-3\gamma)$$

$$= -27\alpha\beta\gamma \quad \text{(using 1)}$$

$$= 27q \quad \text{(using 3)}$$

The required equation is $x^3 - S_1x^2 + S_2x - S_3 = 0$.

$x^3 + 9px - 27q = 0$ which is the required equation.

are in arithmetic progression.

Soln:

Since the roots are in A.P.

Let us take them to be $a-d$, a, $a+d$.

$$S_1 = (a-d) + a + (a+d) = 6$$

$$3a = 6 \Rightarrow a = 2$$

$$S_3 = (a-d)a(a+d) = -\frac{18}{4} = -\frac{9}{2}$$

$$a(a^2-d^2) = -\frac{9}{2}$$

$$2(4-d^2) = -\frac{9}{2}$$

$$4-d^2 = \frac{-9}{4}$$

$$d^2 = 4+\frac{9}{4} = \frac{25}{4}$$

$d = \frac{5}{2}$ (taking positive value only)

The roots are $\frac{-1}{2}, 2, \frac{9}{2}$

VIP Problem 10:

**

Show that the roots of the equation $px^3+qx^2+rx+s=0$ in G.P. iff $r^3p = q^3s$.

Soln:

The roots be $\frac{a}{r}$, a, ar

$$S_1 = \frac{a}{r} + a + ar = -\frac{q}{p} \longrightarrow (1)$$

$$S_3 = \frac{a}{r} a \, ar = -\frac{s}{p} \longrightarrow ③$$

From ③ $\Rightarrow a^3 = -\frac{s}{p} \longrightarrow ④$

Dividing ② by ① $\Rightarrow$

$$\frac{a^2\left(\frac{1}{r}+r+1\right)}{a\left(\frac{1}{r}+r+1\right)} = -\frac{r}{q}$$

$$a = -\frac{r}{q} \longrightarrow ⑤$$

Substuting ⑤ in ④

$$\left(-\frac{r}{q}\right)^3 = -\frac{s}{p}$$

$$r^3 p = q^3 s.$$

conversely, suppose $r^3 p = q^3 s \longrightarrow ⑥$

$a = \frac{-r}{q}$ is a root of the given equation.

consider $p\left(\frac{-r}{q}\right)^3 + q\left(\frac{-r}{q}\right)^2 + r\left(\frac{-r}{q}\right) + s$

$$= -\frac{pr^3}{q^3} + \frac{r^2}{q} - \frac{r^2}{q} + s$$

using ⑥

$$= \frac{-pr^3 + q^3 s}{q^3} = \frac{-pr^3 + pr^3}{q^3} = 0$$

$x + \left(\frac{r}{q}\right)$ is a factor.

Dividing the given equation by $\left(x + \frac{r}{q}\right)$

$$q(x) = px^2 + \left(q - \frac{pr}{q}\right)x + \frac{pr^2}{q^2}.$$

Let α and β be the ~~roots~~ roots of $q(x) = 0$.

The product of the roots $\alpha\beta = \frac{r^2}{q^2} = \left(-\frac{r}{q}\right)^2 = a^2$

α, a, β are in G.P.

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problem 11.

If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two prove that $p^3 + 8r = 4pq$.

Soln:

Let $\alpha, \beta, \gamma, \delta$ be the roots of the equations.

$S_1 = \alpha + \beta + \gamma + \delta = -p \quad \rightarrow (1)$

$S_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q \quad \rightarrow (2)$

$S_3 = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha = -r \quad \rightarrow (3)$

Also given $\alpha + \beta = \gamma + \delta \quad \rightarrow (4)$

From (1) and (4) $\Rightarrow$ $\alpha + \beta = -\frac{1}{2}p \quad \rightarrow (5)$

From ② $\alpha\beta + \alpha(\gamma+\delta) + \beta(\gamma+\delta) + \gamma\delta = q$

$$\alpha\beta + (\alpha+\beta)(\gamma+\delta) + \gamma\delta = q$$

$$\alpha\beta + \left(\frac{-p}{2}\right)\left(\frac{-p}{2}\right) + \gamma\delta = q$$

$$\alpha\beta + \frac{1}{4}p^2 + \gamma\delta = q$$

$$\alpha\beta + \gamma\delta = q - \frac{1}{4}p^2 \quad \longrightarrow ⑥$$

From ③ $\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -r$

$$(\alpha\beta + \gamma\delta)(\alpha+\beta) = -r$$

$$(\alpha\beta + \gamma\delta)\left(\frac{-p}{2}\right) = -r$$

$$\alpha\beta + \gamma\delta = \frac{2r}{p} \quad \longrightarrow ⑦$$

From ⑥ and ⑦ $\quad q - \frac{1}{4}p^2 = \frac{2r}{p}$

$$4pq = p^3 + 8r$$

---

problem 12 :

If the product of two roots of $x^4 + px^3 + qx^2 + rx + s = 0$ is equal to the product of the other two show that $r^2 = p^2 s$.

Soln:

$$f(x) = x^4 + px^3 + qx^2 + rx + s = 0$$

Let $\alpha, \beta, \gamma, \delta$ be the roots of $f(x)=0$

$\Sigma\alpha = -p \longrightarrow ①$

$\Sigma\alpha\beta = q \longrightarrow ②$

$\Sigma\alpha\beta\gamma = -r \longrightarrow ③$

$\alpha\beta\gamma\delta = s \longrightarrow ④$

Also given that $\alpha\beta = \gamma\delta \longrightarrow ⑤$

From ④ and ⑤ $\alpha\beta = \gamma\delta = \sqrt{s} \longrightarrow ⑥$

$$f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

$$= [x^2 - (\alpha+\beta)x + \alpha\beta][x^2 - (\gamma+\delta)x + \gamma\delta]$$

$$= (x^2 - \lambda x + \sqrt{s})(x^2 - \mu x + \sqrt{s})$$

where $\lambda = \alpha+\beta$ and $\mu = \gamma+\delta$

$$f(x) = x^4 - (\lambda+\mu)x^3 + (\lambda\mu + 2\sqrt{s})x^2 - \sqrt{s}(\lambda+\mu)x + s$$

Equating the coefficients of x^3 and x on both sides

$\lambda+\mu = -p \qquad \lambda+\mu = -\frac{r}{\sqrt{s}}$

$$-p = -\frac{r}{\sqrt{s}} \Rightarrow p^2 s = r^2.$$

Find the condition that the equation $x^4+px^3+qx^2+rx+s$ should have two roots α, β connected by the relation $\alpha+\beta=0$.

Soln:

$$f(x) = x^4+px^3+qx^2+rx+s.$$

Let $\alpha, \beta, \gamma, \delta$ be the roots of $f(x)=0$

Given $\alpha+\beta=0. \rightarrow (1)$

Also we have $\underbrace{\alpha+\beta}_{0}+\gamma+\delta=-p \rightarrow (2)$

$$0+\gamma+\delta=-p \rightarrow (3)$$

$$f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

$$= [x^2-(\alpha+\beta)x+\alpha\beta][x^2-(\gamma+\delta)x+\gamma\delta]$$

$$= (x^2+\lambda)(x^2+px+\mu).$$

where $\lambda=\alpha\beta$ and $\mu=\gamma\delta$

$$= x^4+px^3+(\lambda+\mu)x^2+p\lambda x+\lambda\mu.$$

Equating the coefficients of x^2, x and the constant term

$$\lambda+\mu=q \rightarrow (4)$$

$$p\lambda=r \rightarrow (5)$$

$$\lambda\mu=s \rightarrow (6)$$

From (5) $\lambda=\frac{r}{p} \rightarrow (7)$

From ⑥ $\mu = \frac{s}{\lambda} = \frac{sp}{r}$ $\longrightarrow$ ⑧

Substituting ⑦ and ⑧ in ④.

$$\frac{r}{p} + \frac{sp}{r} = q$$

$$r^2 + sp^2 = pqr$$

This is the required condition

Problem 14 :

Solve the equation $8x^4 - 90x^3 + 315x^2 - 405x + 162 = 0$ given that the roots are in G.P.

Soln:

$$f(x) = 8x^4 - 90x^3 + 315x^2 - 405x + 162$$

Let $\alpha, \beta, \gamma, \delta$ be the roots of $f(x) = 0$

Since the roots are in G.P. $\frac{\beta}{\alpha} = \frac{\gamma}{\beta} = \frac{\delta}{\gamma}$.

$$\alpha\delta = \beta\gamma \longrightarrow ①$$

$$\alpha\beta\gamma\delta = \frac{162}{8} = \frac{81}{4} \longrightarrow ②$$

From ① and ② $\alpha\delta = \beta\gamma = \frac{9}{2} \longrightarrow ③$

$$f(x) = 8(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

$$= 8\left[x^2 - (\alpha+\delta)x + \alpha\delta\right]\left[x^2 - (\beta+\gamma)x + \beta\gamma\right]$$

$$= 8\left(x^2 - px + \frac{9}{2}\right)\left(x^2 - qx + \frac{9}{2}\right)$$

where $p = \alpha + \delta$ and $q = \beta + \gamma$

$$f(x) = 8\left[x^4 - (p+q)x^3 + (pq+9)x^2 - \frac{9}{2}(p+q)x - \frac{81}{4}\right]$$

Equating the coefficients of x^3 and x^2

$$p+q = \frac{90}{8} \longrightarrow (4)$$

$$pq + 9 = \frac{315}{8} \longrightarrow (5)$$

Using (4) in (5)

$$p\left(\frac{45}{4} - p\right) + 9 = \frac{315}{8}$$

$45 \times 2 = 90$

$$\frac{45}{4}p - p^2 + 9 - \frac{315}{8} = 0$$

$315 - 72 = 243$

$$\frac{-8p^2 + 90p + 72 - 315}{8} = 0$$

$243 \times 8 = 1944$

$$-8p^2 + 90p - 243 = 0$$

$$8p^2 - 90p + 243 = 0$$

$$p = \frac{27}{4} \; ; \; \frac{9}{2}$$

The roots of the given equation are the roots of two quaril equations

$$x^2 - \frac{9}{2}x + \frac{9}{2} = 0 \quad \text{and} \quad x^2 - \frac{27}{4}x + \frac{9}{2} = 0$$

The required roots are the roots of the equations $2x^2 - 9x + 9 = 0$ & $4x^2 - 27x + 18 = 0$

$(x-6)(4x-3)=0$, $(x-3)(2x-3)=0$.

Hence the roots are $\frac{3}{4}, \frac{3}{2}, 3, 6$ (which are viously in G.P)

Unit - II

Sum of the powers of the roots:

Notation :

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$.

Let $S_r = \alpha_1^r + \alpha_2^r + \ldots + \alpha_n^r$, where $r \in Z$, denote the sum of r^{th} powers of the roots.

The following theorem due to Newton gives a relation connecting powers of the roots and the coefficients of the equation.

Theorem 7: (Newton's theorem)

* 2M [Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$. Then

(i) $a_0S_r + a_1S_{r-1} + \ldots + a_{r-1}S_1 + ra_r = 0$ if $r \leq n$

(ii) $a_0S_r + a_1S_{r-1} + \ldots + a_{n-1}S_{r-n+1} + a_nS_{r-n} = 0$ if $r > n$]

proof:

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = a_0(x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_n)$$

putting $x = \frac{1}{y}$ and multiplying by y^n on both sides

$$a_0 + a_1 x + \ldots + a_{n-1} y^{n-1} + a_n y^n = a_0(1-\alpha_1 y)(1-\alpha_2 y) \ldots (1-\alpha_n y)$$

Taking logaritham on both sides and differentiating w.r.t y

$$\frac{a_1 + 2a_2 y + 3a_3 y^2 + \ldots + n a_n y^{n-1}}{a_0 + a_1 y + \ldots + a_{n-1} y^{n-1} + a_n y^n} = \left[\frac{-\alpha_1}{1-\alpha_1 y} - \frac{\alpha_2}{1-\alpha_2 y} - \ldots - \frac{\alpha_n}{1-\alpha_n y}\right]$$

$$= -\left[\alpha_1(1-\alpha_1 y)^{-1} + \alpha_2(1-\alpha_2 y)^{-1} + \ldots + \alpha_n(1-\alpha_n y)^{-1}\right]$$

$$= -\left[\alpha_1(1+\alpha_1 y + \alpha_1^2 y^2 + \ldots) + \alpha_2(1+\alpha_2 y + \alpha_2^2 y^2 + \ldots) + \ldots + \alpha_n(1+\alpha_n y + \alpha_n^2 y^2 + \ldots)\right]$$

the expansions in the right side being valid for sufficiently small values

$$= -\left[(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + y(\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2) + \ldots + y^{n-1}(\alpha_1^n + \alpha_2^n + \ldots + a_n^n) + \ldots\right].$$

$$= -\left[S_1 + S_2 y + y^2 S_3 + \dots + y^{n-1} S_n + \dots\right]$$

$$a_1 + 2a_2 y + 3a_3 y^2 + \dots + n a_n y^{n-1}$$

$$= -(S_1 + yS_2 + y^2 S_3 + \dots + y^{n-1} S_n + \dots) \times$$

$$(a_0 + a_1 y + a_2 y^2 + \dots + a_{n-1} y^{n-1} + a_n y^n)$$

Equating the coefficients of y^r

$$r a_r = -(a_0 S_r + a_1 S_{r-1} + \dots + a_{r-1} S_1) \quad \text{if } r \le n.$$

and $$0 = -(a_0 S_r + a_1 S_{r-1} + \dots + a_n S_{r-n}) \quad \text{if } r > n$$

$$a_0 S_r + a_1 S_{r-1} + \dots + a_{r-1} S_1 + r a_r = 0 \quad \text{if } r \le n.$$

and $$a_0 S_r + a_1 S_{r-1} + \dots + a_n S_{r-n} = 0 \quad \text{if } r > n.$$

Remark 1:

To find the value of S_r where r is a positive integer we proceed as follows. we have

$$a_0 S_r + a_1 S_{r-1} + \dots + a_{r-1} S_1 + r a_r = 0 \quad \text{if } r \le n.$$

putting $r = 1, 2, 3, \dots$ we obtain

$$a_0 S_1 = -a_1 \longrightarrow ①$$

$$a_0 S_2 + a_1 S_1 + 2a_2 = 0 \longrightarrow ②$$

$$a_0 S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0 \longrightarrow ③$$

$$a_0 S_4 + a_1 S_3 + a_2 S_2 + a_3 S_1 + 4a_4 = 0 \longrightarrow ④$$

$\dots\dots$

From ① $S_1 = -a\sqrt{a_0}$

using the value of S_1 in ② we can get S_2.

Proceeding like this we obtain S_r for any positive integer r.

Remark 2 :

S_{-r} can be calculated from the fact that S_{-r} for $f(x)$ is the same as S_r for $f(1/x)$.

Theorem 8 :

Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ be the roots of the equation $a_0x^n + a_1 a x^{n-1} + \dots + a_n = 0$.

Then

$$\log\left[1 + \frac{a_1}{a_0}y + \frac{a_2}{a_0}y^2 + \dots + \frac{a_n}{a_0}y^n\right]$$

$$= -S_1 y - \frac{1}{2}S_2 y^2 - \frac{1}{3}S_3 y^3 - \dots$$

where $x = 1/y$.

proof :

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

putting $x = 1/y$ and multiplying by y^n

$$a_0 + a_1y + \dots + a_{n-1}y^{n-1} + a_ny^n = a_0(1-\alpha_1 y)(1-\alpha_2 y)\dots(1-\alpha_n y)$$

Taking logaritham on both sides,

$$\log(a_0 + a_1 y + \dots + a_n y^n) = \log a_0 + \log(1-\alpha_1 y) + \log(1-\alpha_2 y) + \dots + \log(1-\alpha_n y)$$

$$\log\left[a_0\left(1 + \frac{a_1}{a_0} y + \dots + \frac{a_n}{a_0} y^n\right)\right] = \log a_0 + \log(1-\alpha_1 y) + \log(1-\alpha_2 y) + \dots + \log(1-\alpha_n y).$$

$$\log\left[1 + \frac{a_1}{a_0} y + \frac{a_2}{a_0} y^2 + \dots + \frac{a_n}{a_0} y^n\right]$$

$$= \log(1-\alpha_1 y) + \log(1-\alpha_2 y) + \dots + \log(1-\alpha_n y)$$

$$= -\left(\alpha_1 y + \frac{\alpha_1^2 y^2}{2} + \frac{\alpha_1^3 y^3}{3} + \dots\right)$$

$$-\left(\alpha_2 y + \frac{\alpha_2^2 y^2}{2} + \frac{\alpha_2^3 y^3}{3} + \dots\right)$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$-\left(a_n y + \frac{a_n^2 y^2}{2} + \frac{a_n^3 y^3}{3} + \dots\right)$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$= -S_1 y - \frac{1}{2} S_2 y^2 - \frac{1}{3} S_3 y^3 - \dots - \frac{1}{r} S_r y^r - \dots$$

now, expanding $\log\left(1 + \frac{a_1}{a_0} y + \frac{a_2}{a_0} y^2 + \dots + \frac{a_n}{a_0} y^n\right)$ in asscend powers of y and equating the coefficients of

y' we obtain the requir result.

Problem 1: show that the sum of the 6^{th} powers of the roots of $x^7 - x^4 + 1 = 0$ is 3.

Soln:

we have to prove $S_6 = 3$.

$a_0 = 1 \quad a_1 = a_2 = 0 \quad a_3 = -1 \quad a_4 = a_5 = a_6 = 0$

$a_7 = 1$.

By Newton's theorem,

$a_0 S_1 + a_1 = 0 \Rightarrow S_1 = 0$

$a_0 S_2 + a_1 S_1 + 2a_2 = 0 \Rightarrow S_2 = 0$

$a_0 S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0 \Rightarrow S_3 = 3$

$a_0 S_4 + a_1 S_3 + a_2 S_2 + a_3 S_1 + 4a_4 = 0 \Rightarrow S_4 - S_1 = 0$

$S_4 = 0$

$a_0 S_5 + a_1 S_4 + a_2 S_3 + a_3 S_2 + a_4 S_1 + 5a_5 = 0 \Rightarrow S_5 - S_2 = 0$

$S_5 = 0$

$a_0 S_6 + a_1 S_5 + a_2 S_4 + a_3 S_3 + a_4 S_2 + a_5 S_1 + 6a_6 = 0$

$\Rightarrow S_6 - S_3 = 0$

Show that the sum of cubes of the roots of

$x^n + p_1 x^{n-1} + \dots + p_n = 0$ is $3p_1p_2 - p_1^3 - 3p_3$

<u>Soln</u>:

To prove $S_3 = 3p_1p_2 - p_1^3 - 3p_3$

$a_0 = 1$, $a_1 = p_1$, $a_2 = p_2$, ..., $a_n = p_n$.

By Newton's theorem,

$a_0 S_1 + a_1 = 0 \Rightarrow S_1 + p_1 = 0$

$S_1 = -p_1$

$a_0 S_2 + a_1 S_1 + 2a_2 = 0 \Rightarrow S_2 - p_1^2 + 2p_2 = 0$

$S_2 = p_1^2 - 2p_2$

$a_0 S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0$

$S_3 + p_1(p_1^2 - 2p_2) + p_2(-p_1) + 3p_3 = 0$

$S_3 = p_1p_2 - p_1^3 + 2p_1p_2 - 3p_3$

$S_3 = 3p_1p_2 - p_1^3 - 3p_3$

Problem 3:

Show that the sum of the m^{th} powers of the roots of

$x^n - 2x^{n-1} - 2x^{n-2} - \dots - 2x - 2 = 0$ is $3^m - 1$ where $m \le n$.

Soln:

To prove $S_m = 3^m - 1$.

Prove the result of induction on m.

$a_0 = 1, \quad a_1 = a_2 = \dots = a_m = -2$ Since $m \leq n$

By Newton's theorem for $m = r$

$$a_0 S_m + a_1 S_{m-1} + a_2 S_{m-2} + \dots + a_{m-1} S_1 + m a_m = 0$$

$$S_m = 2(S_1 + S_2 + \dots + S_{m-1} + m) \longrightarrow (1)$$

putting $m = 1$,

$$S_1 = 2 = 3 - 1$$

now assume that the result is true for all

$r < m$.

From (1) $\Rightarrow$ $S_m = 2\left[(3-1) + (3^2-1) + \dots + (3^{m-1} - 1) + m\right]$

$$= 2\left[3 + 3^2 + \dots + 3^{m-1} - (m-1) + m\right]$$

$$= 2\left[\frac{3(3^{m-1} - 1)}{3-1} + 1\right]$$

$$= 3(3^{m-1} - 1) + 2$$

$$= 3^{m-1} - 1$$

Hence by induction the result is true for all m.

Problem 4 :

With the usual notation for the equation $x^4+px^3+qx+r=0$ prove that $s_n+ps_{n-1}+qs_{n-3}+rs_{n-4}=0$.

Soln:

$$x^4+px^3+qx+r=0.$$

Let α_1, α_2, α_3, α_4 be the roots of this equation and

Multiplying the given equation by x^{n-4}

$$x^n+px^{n-1}+qx^{n-3}+rx^{n-4}=0$$

Substituting $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in this equation and adding $S_n+pS_{n-1}+qS_{n-3}+rS_{n-4}=0$

Problem 5 :

Find (i) $\Sigma\alpha^2$ (ii) $\Sigma\alpha^{-2}$ for the equation

$x^4-x^3-19x^2+49x-30=0$

Soln:

With the usual notation. we have to find S_2 and S_{-2} for $x^4-x^3-19x^2+49x-30=0 \rightarrow (1)$

$a_0=1$ $a_1=-1$ $a_2=-19$ $a_3=49$ $a_4=-30$

(i) By Newton's theorem,

$$a_0 s_1 + a_1 = 0 \Rightarrow s_1 = 1.$$

$$a_0 s_2 + a_1 s_1 + 2a_2 = 0 \Rightarrow s_2 - 1 - 38 = 0$$

$$s_2 = 39$$

(ii) To find S_{-2}

put $x = 1/y$ in ①

$$(1/y)^4 - (1/y)^3 - 19(1/y)^2 + 49(1/y) - 30 = 0$$

$$\frac{1}{y^4} - \frac{1}{y^3} - \frac{19}{y^2} + \frac{49}{y} - 30 = 0$$

$$\frac{1 - y - 19y^2 + 49y^3 - 30y^4}{y^4} = 0$$

$$30y^4 - 49y^3 + 19y^2 + y - 1 = 0$$

S_2 of the equation is S_{-2} of the give equation

$a_0 = 30 \quad a_1 = -49 \quad a_2 = 19 \quad a_3 = 1 \quad a_4 = -1.$

By Newton's Theorem,

$$a_0 s_1 + a_1 = 0 \Rightarrow 30 s_1 = 49 \Rightarrow s_1 = \frac{49}{30}$$

$$a_0 s_2 + a_1 s_1 + 2a_2 = 0 \Rightarrow 30 s_2 - 49\left(\frac{49}{30}\right) + 38 = 0$$

$$30 s_2 = \frac{49^2}{30} - 38 = \frac{1261}{30}$$

$$S_2 = \frac{1261}{30 \times 30} = \frac{1261}{900}$$

Hence $S_{-2} = \frac{1261}{900}$

Problem 6:

If $\alpha+\beta+\gamma = 6$, $\alpha^2+\beta^2+\gamma^2 = 14$ and $\alpha^3+\beta^3+\gamma^3 = 36$ prove that $\alpha^4+\beta^4+\gamma^4 = 98$.

Soln:

Let the roots are α, β, γ.

$$x^3+px^2+qx+r=0 \rightarrow (1)$$

$S_1 = 6 \quad S_2 = 14 \quad S_3 = 36$

We have to prove $S_4 = 98$.

$a_0 = 1 \quad a_1 = p \quad a_2 = q \quad a_3 = r$

By Newton's Theorem.

$$a_0 S_1 + a_1 = 0 \Rightarrow 6 + p = 0 \Rightarrow p = -6$$

$$\begin{array}{r} 14 \\ 6 \\ \hline 84 \end{array}$$

$$a_0 S_2 + a_1 S_1 + 2a_2 = 0 \Rightarrow 14 + 6p + 2q = 0$$

$$14 - 36 + 2q = 0$$

$$+2q = 22$$

$$q = 11$$

$$a_0 S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0 \Rightarrow 36 + (-6)14 + (11)6 + 3(r) = 0$$

$36 - 84 + 66 + 3\gamma = 0$

$18 + 3\gamma = 0$

$3\gamma = -18 \Rightarrow \gamma = \frac{-18}{3} = -6.$

$\gamma = -6.$

The equation ① becomes $x^3 - 6x^2 + 11x - 6 = 0$

$a_0 = 1 \quad a_1 = -6 \quad a_2 = 11 \quad a_3 = -6$

$a_0 S_4 + a_1 S_3 + a_2 S_2 + a_3 S_1 + 4a_4 = 0$

$S_4 - 6 \times 36 + 11 \times 14 - 6 \times 6 = 0$

$S_4 = 216 - 154 - 36 = 98$

Hence $\alpha^4 + \beta^4 + \gamma^4 = 98.$

Problem 7 :

Find S_{16} for the equation $x^8 + ax + b = 0$

Soln:

Let $\alpha_1, \alpha_2, \ldots \alpha_8$ be the roots of the given equation.

$x^8 = -(ax + b)$

$x^{16} = a^2x^2 + 2abx + b^2 \rightarrow ①$

Substituting $x = \alpha_1, \alpha_2, \ldots, \alpha_8$ in ① and adding

$S_{16} = a^2 S_2 + 2ab S_1 + 8b^2 \rightarrow ②$

find S_1 and S_2 for the equation $x^8 + ax + b$

$a_0 = 1 \quad a_1 = a_2 = a_3 = a_4 = \dots = a_6 = 0 \quad a_7 = a \quad a_8 =$

By Newton's Theorem,

$$a_0 S_1 + a_1 \Rightarrow S_1 = 0$$

$$a_0 S_2 + a_1 S_1 + 2a_2 = 0 \Rightarrow S_2 = 0$$

From ② $\Rightarrow$ $S_{16} = 0 + 0 + 8b^2$

$$S_{16} = 8b^2$$

Problem 8:

If $\alpha + \beta + \gamma = 0$ prove $\frac{\Sigma\alpha^7}{7} = \frac{\Sigma\alpha^5}{5} \times \frac{\Sigma\alpha^2}{2}$

Soln:

Let α, β, γ be the roots of the given equation.

$x^3 + P_1 x^2 + P_2 x + P_3 = 0$ given $\Sigma\alpha_1 = S_1 = 0 \longrightarrow$ ①

The equation is $x^3 + P_2 x + P_3 = 0 \longrightarrow$ ②

To prove $\frac{S_7}{7} = \frac{S_5}{5} \times \frac{S_2}{2}$ for ②.

$a_0 = 1 \quad a_1 = 0 \quad a_2 = P_2 \quad a_3 = P_3$

By Newton's Theorem,

$S_2 = -2P_2 \quad S_3 = -3P_3 \quad S_4 = 2P_2^2 \quad S_5 = 5P_2P_3 \quad S_6 = 3P_3^2 - 2P_2^3.$

$\therefore$ and $S_7 = -7P_2^2 P_3$

$\frac{S_7}{7} = -P_2^2 P_3$

$\frac{S_7}{7} = \frac{S_5}{5} \times \frac{S_2}{2}$

Aliter:

Putting $x = 1/y$ in $x^3 + P_2 x^2 + P_3 = 0$ it we get into

$1 + P_2 y^2 + P_3 y^3 = 0.$

S_2 is the coefficient of y^2 in

$$-2 \log (1 + P_2 y^2 + P_3 y^3) \rightarrow (1)$$

S_5 is the coefficient of y^5 in $-5 \log (1 + P_2 y^2 + P_3 y^3) \rightarrow (2)$

S_7 is the coefficient of y^7 in $-7 \log (1 + P_2 y^2 + P_3 y^3) \rightarrow (3)$

consider the third equation.

$$-7 \log (1 + P_2 y^2 + P_3 y^3) = -7 \log [1 + y^2 (P_2 + P_3 y)]$$

$$= -7 \left[y^2 (P_2 - P_3 y) - \frac{y^4 (P_2 + P_3 y)^2}{2} + \frac{y^6 (P_2 + P_3 y)^3}{3} \cdots\cdots \right]$$

$$S_7 = \text{coefficient of } y^7 = -7 \left[\frac{3P_2^2 P_3}{3} \right] = -7P_2^2 P_3$$

Similarly from (2) and (1)

$$S_5 = \text{coefficient of } y^5 = -5 \left[\frac{-P_2 P_3}{2} \right] = 5 P_2 P_3$$

$S_2 =$ coefficient of $y^2 = -2P_2$

$$\frac{S_5}{5} \times \frac{S_2}{2} = \frac{5P_2P_3}{5} \times \frac{-2P_2}{2} = -P_2^2 P_3$$

$$\frac{S_7}{7} = \frac{-7P_2^2P_3}{7} = -P_2^2 P_3$$

$$\frac{S_7}{7} = \frac{S_5}{5} \times \frac{S_2}{2}$$

Problem 9:

Show that the sum of the 20^{th} powers of the roots of equation $x^4+ax+b=0$ is $50a^4b^2-4b^5$.

Soln:

To find S_{20} for the equation

put $x = 1/y$ in the given equation then it becomes

$1+ay^3+by^4=0$

$$S_{20} = \text{coefficient of } y^{20} \text{ in } -20\log(1+ay^3+by^4)$$

$$= \text{coefficient of } y^{20} \text{ in } -20\log\left[1+y^3(a+by)\right]$$

$$= \text{coefficient of } y^{20} \text{ in } -20\left[y^3(a+by) - \frac{y^6(a+by)^2}{2} + \frac{y^9(a+by)^3}{3} + - \right.$$

$$\left. \frac{y^{12}(a+by)^4}{4} + \frac{y^{15}(a+by)^5}{5} - \frac{y^{18}(a+by)^6}{6} + \cdots\right]$$

$$= -20\left[\frac{b^5}{5} - \frac{1}{6} \times {}^6C_2\, a^4b^2\right] = 50a^4b^2 - 4b^5.$$

Reciprocal EQUATIONS:

Definition :

The equation $f(x)=0$ is called a reciprocal equation (R.E) whenever α is a root of the equation $\frac{1}{\alpha}$ is also a root.

Thus if $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are the roots of a R.E then the same numbers $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots\ldots, \frac{1}{\alpha_n}$ are the same as $\alpha_1, \alpha_2, \ldots, \alpha_n$ in some order.

Examples :

(i) $x^2+2x+1=0$ is a RE and its roots are -1 and -1

(ii) $2x^2-5x+2=0$ is a R.E and its roots are 2 and $\frac{1}{2}$

(iii) $2x^3+3x^2-3x-2=0$ is a R.E and its roots are $-2, -\frac{1}{2}$ and 1

Theorem 9:

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $f(x)=0$, then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are the roots of $x^n f\left(\frac{1}{x}\right)=0$.

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $f(x)=0$

we have $f(\alpha_i)=0 \quad = 1, 2$

Let $g(x) = x^n f\left(\frac{1}{x}\right)$.

Then $g\left(\frac{1}{\alpha_i}\right) = \left(\frac{1}{\alpha_i}\right)^n f(\alpha_i) = 0$

hence $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are the roots of $g(x)=0$.

Hence the result.

Remark: If $f(x)=0$ is a R.E then $f(x)=0$ and $x^n f\left(\frac{1}{x}\right)$ both have the same roots.

Theorem 10:

$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$ is a R.E. If and only if $a_{n-1} = \pm a_r$ where $(0 \leq r \leq n)$.

proof:

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n.$$

Hence $x^n f\left(\frac{1}{x}\right) = x^n\left[a_0\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + \ldots + a_n\right]$

$$= a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0.$$

Now suppose $f(x)=0$ is a R.E. Then $f(x)=0$ and $x^n f\left(\frac{1}{x}\right)=0$ both have the same roots and hence the corresponding coefficients of the two equations are proportional.

$$\therefore \frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \cdots\cdots = \frac{a_r}{a_{n-r}} = \cdots\cdots = \frac{a_n}{a_0} = k$$

Since $\frac{a_0}{a_n} = \frac{a_n}{a_0} = k$ we get $k^2 = 1 \Rightarrow k = \pm 1$

$$\frac{a_r}{a_{n-r}} = \pm 1 \Rightarrow a_r = \pm a_{n-r}$$

conversely suppose that $a_r = \pm a_{n-r}$. Then the equation $f(x) = 0$ $x^n f\left(\frac{1}{x}\right) = 0$ are same and hence $f(x) = 0$ is a R.E.

Definition:

A R.E $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots\cdots + a_0 = 0$ is said to be a type if $a_{n-r} = a_r$ and is said to be of second type of if $a_{n-r} = -a_r$.

Remark: $f(x) = 0$ is a R.E of first type if and only if $f(x) = x^n f\left(\frac{1}{x}\right)$ and is a R.E of second type if and only if $f(x) = -x^n f\left(\frac{1}{x}\right)$.

Definition :

A reciprocal equation $f(x)=0$ is called a standard reciprocal equation (S.R.E) if it is of first type and the degree of $f(x)$ is even.

Examples :

(i) $4x^4-20x^3+33x^2-20x+4=0$ is a S.R.E.

(ii) $6x^5+11x^4-33x^3-33x^2+11x+6=0$ is a R.E of ~~first~~ first type and odd degree.

(iii) $x^5-5x^4+9x^3-9x^2+5x-1=0$ is a R.E of second type and odd degree.

Any RE can be reduced to a S.R.E on division by a suitable factor shown in the following theorems.

Theorem 11:

If $f(x)=0$ is a R.E of first type and odd degree then (i) $x+1$ is a factor of $f(x)$ and

(ii) $\dfrac{f(x)}{x+1}$ is a S.R.E.

proof :

$f(x)=a_0x^{2n+1}+a_1x^{2n}+\dots+a_{2n+1}=0$ be a R.E of first type and of odd degree.

$$a_{2n+1-r} = a_r \longrightarrow (1)$$

(i) Now, $f(-1) = -a_0 + a_1 - a_2 + \dots + (-1)^{n+1} a_n + (-1)^{n+2} a_{n+1}$

$$+ \dots + (-1)^{2n+2} a_{2n+1}$$

$$= (a_{2n+1} - a_0) + (a_1 - a_{2n}) + (a_{2n-1} - a_2) + \dots + \left[(-1)^{n+1} a_n + (-1)^{n+2} a_{n+1}\right]$$

$$= 0 \quad [\text{by } (1)]$$

Hence $x+1$ is a factor of $f(x)$.

(ii) Let $\varphi(x) = \dfrac{f(x)}{x+1}$. Clearly $\varphi(x)$ is a polynomial of degree $2n$.

$$x^{2n} \varphi\left(\frac{1}{x}\right) = x^{2n}\left[\frac{f(1/x)}{(1/x)+1}\right] = \frac{x^{2n+1} f(1/x)}{x+1}.$$

$$= \frac{f(x)}{x+1} \quad \left[\text{Since } f(x) \text{ is a R.E of first type}\right]$$

$$= \varphi(x)$$

Hence $\varphi(x) = 0$ is a R.E of first type and of even degree.

$\varphi(x) = 0$ is a S.R.E .

Theorem 12:

If $f(x)=0$ is a RE of second type and of odd degree, then $x-1$ is a factor of $f(x)$ and $\frac{f(x)}{x-1}$ is a S.R.E.

proof:

Similar to that of Theorem 11.

Theorem 13:

If $f(x)=0$ is a R.E of second type and even degree, then is a factor of $f(x)$ and $\frac{f(x)}{x^2-1}$ is a S.R.E.

proof:

Let $f(x) = a_0 x^{2n} + a_1 x^{2n-1} + \dots + a_{2n}$ be a R.E of second type. Hence $a_{2n-r} = -a_r \rightarrow ①$

putting $r=n$. $a_n = -a_n$

Hence $2a_n = 0 \Rightarrow a_n = 0$

Now, $f(1) = a_0 + a_1 + a_2 + \dots + a_{n-1} + a_{n+1} + \dots + a_{2n}$.

$= (a_0 + a_{2n}) + (a_1 + a_{2n-1}) + \dots + (a_{n-1} + a_{n+1})$

$= 0$ [by ①]

Hence $x-1$ is a factor of $f(x)$.

Also,

$$f(-1) = a_0 - a_1 + a_2 + \dots + (-1)^{n-1} a_{n-1} + (-1)^{n+1} a_{n-1} + \dots + (-1)^{n-}$$

$$= (a_0 + a_{2n}) - (a_1 + a_{2n-1}) + \dots + (-1)^{n-1}(a_{n-1} + a_{n+1})$$

$$= 0 \quad [\text{by } ①]$$

Hence $x+1$ is a factor of $f(x)$ so that

$(x-1)(x+1) = x^2-1$ is a ? of $f(x)$.

Now, let $\varphi(x) = \dfrac{f(x)}{x^2-1}$.

clearly $\varphi(x)$ is a polynomial of degree $2n-2$ (even)

Now $x^{2n-2}\varphi\left(\frac{1}{x}\right) = \dfrac{x^{2n-2} f(1/x)}{(1/x^2)-1} = \dfrac{x^{2n} f(1/x)}{1-x^2}$

$= \dfrac{-f(x)}{1-x^2}$. [Since $f(x)$ is a R.E of second type].

Hence $\varphi(x) = 0$ is a R.E of first type and even

degree $\varphi(x) = 0$ is a S.R.E.

ie. now give a method of solving a S.R.E

Remark :

From the above theorems it follows that the problem of solving a R.E reduces to that of solving a S.R.E.

now give a method of solving a SRE

Let $f(x) = a_0 x^{2n} + a_1 x^{2n-1} + \dots + a_{2n} = 0$ be a SRE

So that $a_{2n-r} = a_r$.

$$\therefore f(x) = a_0(x^{2n}+1) + a_1(x^{2n-1}+x) + \dots + a_{n-1}(x^{n+1}+x^{n-1}) + a_n$$

$$\frac{f(x)}{x^n} = a_0\left(x^n + \frac{1}{x^n}\right) + a_1\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + \dots + a_{n-1}\left(x + \frac{1}{x}\right) + a_n$$

putting $x + \frac{1}{x} = y$, we obtain a polynomial $g(y)$ of degree n.

Let $y_1, y_2, \dots, y_n$ be the roots of the equation $g(y) = 0$.

$x + \frac{1}{x} = y_i$ gives $x^2 - xy_i + 1 = 0$ and hence for each value of y_i, we get two roots of the given S.R.E.

Solved Problems:

CLP - certified Leisure professional

Problem 1:

Show that $4(x^2-x+1)^3 = 27x^2(x-1)^2$ is a standard reciprocal equation.

Soln:

Let $f(x) = 4(x^2-x+1)^3 = 27x^2(x-1)^2$

$$f\left(\frac{1}{x}\right) = 4\left(\frac{1}{x^2} - \frac{1}{x} + 1\right)^3 - 27\left(\frac{1}{x^2}\right)\left(\frac{1}{x} - 1\right)^2$$

$$= \frac{4(1-x+x^2)^3}{x^6} - \frac{27(1-x)^2}{x^4}$$

$$x^6 f\left(\frac{1}{x}\right) = 4(1-x+x^2)^3 - 27x^2(1-x)^2.$$

$$= 4(x^2-x+1)^3 - 27x^2(x-1)^2$$

$$= f(x).$$

Also $f(x)$ is of degree 6

$\therefore$ $f(x)$ is a S.R.E.

Problem 2:

If 2 and 3 are the roots of the equation

$6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$ find the remaining roots.

odd second.

Solution :

The given equation is a reciprocal equation of second type and 2 degree.

x^2-1 is a factor and hence $1, -1$ are two roots.

Since $2, 3$ are the roots of the given reciprocal equation $\frac{1}{2}, \frac{1}{3}$ are the roots.

the six roots are $1, -1, \frac{1}{2}, 2, 3, \frac{1}{3}$.

Problem 3 :

Solve $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$

Soln :

The given equation is a S.R.E.

$$4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0 \longrightarrow ①$$

Dividing the equation by x^2 and regrouping.

$$4\left(x^2 + \frac{1}{x^2}\right) - 20\left(x + \frac{1}{x}\right) + 33 = 0 \longrightarrow ②$$

put $x + \frac{1}{x} = y \qquad x^2 + \frac{1}{x^2} = y^2 - 2.$

from ② $\Rightarrow \quad 4(y^2-2) - 20y + 33 = 0.$

$$4y^2 - 20y + 25 = 0.$$

$$(2y-5)^2=0$$

$$y=\frac{5}{2},\frac{5}{2}$$

$$x+\frac{1}{x}=\frac{5}{2} \Rightarrow 2x^2-5x+2=0$$

$$(2x-1)(x-2)=0$$

$$x=\frac{1}{2},2.$$

The root of the given equation are $2,\frac{1}{2},2,\frac{1}{2}$

Problem 4 :

Solve $\overset{\text{even}}{6x^4-25x^3}+\overset{\text{first}}{37x^2}-25x+6=0 \longrightarrow (1)$

Soln:

This is a standard reciprocal equation.

Dividing the equation by x^2 and regrouping

$$6\left(x^2+\frac{1}{x^2}\right)-25\left(x+\frac{1}{x}\right)+37=0$$

put $x+\frac{1}{x}=y$ $\quad x^2+\frac{1}{x^2}=y^2-2$

(1) becomes $6(y^2-2)-25y+37=0$

$$6y^2-25y+25=0$$

$$(2y-5)(3y-5)=0$$

$$y=\frac{5}{2},\frac{5}{3}.$$

$$x + \frac{1}{x} = \frac{5}{2} \Rightarrow 2x^2 - 5x + 2 = 0$$

$$(2x-1)(x-2) = 0$$

$$x = 2, \frac{1}{2}$$

$$x + \frac{1}{x} = \frac{5}{3} \Rightarrow 3x^2 - 5x + 3 = 0$$

$$x = \frac{5 \pm \sqrt{25-36}}{6}$$

$$x = \frac{5 \pm i\sqrt{11}}{6}$$

The roots of the given equation are $2, \frac{1}{2}$,

$$\frac{5 + i\sqrt{11}}{6}, \frac{5 - i\sqrt{11}}{6}.$$

Problem 6:

odd first

Solve $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$

<u>Soln:</u>

Let $f(x) = 6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$

We notice that the given equation is a reciprocal equation of first type and of odd degree.

$x+1$ is a factor of $f(x)$.

By actual division

$$\begin{array}{r|rrrrrr} -1 & 6 & 1 & -43 & -43 & 1 & 6 \\ & \downarrow & -6 & 5 & 38 & 5 & -6 \\ \hline & 6 & -5 & -38 & -5 & 6 & \boxed{0} \end{array}$$

$f(x) = (x+1)(6x^4 - 5x^3 - 38x^2 - 5x + 6)$

Now $6x^4 - 5x^3 - 38x^2 - 5x + 6$ is a standard reciprocal equation

Dividing by x^2 and regrouping.

$$6\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) - 38 = 0$$

putting $x^2 + \frac{1}{x^2} = y^2 - 2$ $\quad x + \frac{1}{x} = y$.

$$6(y^2 - 2) - 5y - 38 = 0$$

$$6y^2 - 5y - 50 = 0$$

$$2y(3y - 10) + 5(3y - 10) = 0$$

$$(2y + 5)(3y - 10) = 0$$

$$y = \frac{10}{3} \text{ or } y = \frac{-5}{2}.$$

$$y = \frac{10}{3} \Rightarrow x + \frac{1}{x} = \frac{10}{3}$$

$$3x^2 - 10x + 3 = 0$$

$$(x-3)(3x-1) = 0$$

$$x = 3, \frac{1}{3}$$

$$y = \frac{-5}{2} \Rightarrow x + \frac{1}{x} = -\frac{5}{2}$$

$$2x^2 + 5x + 2 = 0$$

$$(2x+1)(x+2)=0$$

$$x = \frac{-1}{2}, -2$$

The roots of $f(x)$ are $-1, 3, \frac{1}{3}, -2, \frac{-1}{2}$.

Problem 7:

Solve $f(x) = 2\overset{\text{odd}}{x^5} - 15x^4 + 37x^3 - 37\overset{\text{second}}{x^2} + 15x - 2 = 0$

Soln:

We notice that the given equation is a reciprocal equation of second type and of odd degree

$x-1$ is a factor of $f(x)$. By actual division

$$\begin{array}{r|rrrrrr} +1 & 2 & -15 & 37 & -37 & 15 & -2 \\ & \downarrow & 2 & -13 & 24 & -13 & 2 \\ \hline & 2 & -13 & 24 & -13 & 2 & \boxed{0} \end{array}$$

$$f(x) = (x-1)(2x^4 - 13x^3 + 24x^2 - 13x + 2)$$

$2x^4 - 13x^3 + 24x^2 - 13x + 2$ is a E.R.E

Dividing by x^2 and regrouping,

$$2\left(x^2 + \frac{1}{x^2}\right) - 13\left(x + \frac{1}{x}\right) + 24 = 0$$

putting $y = x + \frac{1}{x}$ $\quad y^2 - 2 = x^2 + \frac{1}{x^2}$

$$2y^2 - 4 - 13y + 24 = 0$$

$2y^2 - 13y + 20 = 0$ $\quad$ 40; $\frac{-5}{2} \mid \frac{-8}{2}$

$(y - \frac{5}{2})(y - \frac{8}{2}) = 0$

$y = 4, \frac{5}{2}$

$y = 4 \Rightarrow x + \frac{1}{x} = 4$ $\quad$ $\frac{x^2+1}{x} = 4$

$-4x + x^2 + 1 = 0$ $\quad$ $4x = x^2 + 1$ ee

$x^2 - 4x + 1 = 0$

$x = \frac{4 \pm \sqrt{16-4}}{2}$ $\quad$ $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$x = \frac{4 \pm \sqrt{12}}{2}$

$$\begin{array}{r|l} 2 & 12 \\ 2 & 6 \\ 3 & 3 \\ & 1 \end{array}$$

$x = \frac{4 \pm 2\sqrt{3}}{2}$

$x = \frac{\cancel{2}(2 \pm \sqrt{3})}{\cancel{2}}$

$x = 2 + \sqrt{3},\ 2 - \sqrt{3}.$

$y = \frac{5}{2} \Rightarrow x + \frac{1}{x} = \frac{5}{2}$ $\quad$ $x + \frac{1}{x} - \frac{5}{2} = 0$

$2x^2 - 5x + 2 = 0$ $\quad$ $\frac{2x^2 + 2 - 5x}{2x} = 0$

$x = \frac{1}{2}, 2$ $\quad$ $2x - 5x + 2 = 0$

The roots of $f(x) = 0$ are $1, 2, \frac{1}{2}, 2+\sqrt{3}$ and $2-\sqrt{3}$

Problem 8:

Solve $f(x) = 3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$

Soln:

the given equation is a reciprocal equation of second type and of even degree

$x^2 - 1$ is a factor of $f(x)$. By actual division

1	3	1	-27	0	27	-1	-3	
	↓	3	4	-23	-23	4	3	
-1	3	4	-23	-23	4	3		0
	↓	-3	-1	24	-1	-3		
	3	1	-24	1	3	0		

$$f(x) = (x^2 - 1)(3x^4 + x^3 - 24x^2 + x + 3).$$

$3x^4 + x^3 - 24x^2 + x + 3 = 0$ is a S.R.E.

Dividing by x^2 and regrouping.

$$3\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 24 = 0$$

$$x + \frac{1}{x} = y \qquad x^2 + \frac{1}{x^2} = y^2 - 2$$

$$3y^2 - 6 + y - 24 = 0$$

$$3y^2 + y - 30 = 0$$

$$(3y+10)(y-3) = 0$$

$$y = 3, \frac{-10}{3}$$

$$y = 3 \Rightarrow \quad x + \frac{1}{x} = 3$$

$$x^2 + 1 - 3x = 0$$

$$x^2 - 3x + 1 = 0$$

$$x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$x = \frac{3 \pm \sqrt{5}}{2}$$

$$y = \frac{-10}{3} \Rightarrow \quad x + \frac{1}{x} = \frac{-10}{3}$$

$$3x^2 + 3 = -10x$$

$$3x^2 + 10x + 3 = 0$$

$$9$$

$$\frac{1}{3} \Big| \frac{9}{3}$$

$$x = -3, \frac{-1}{3}$$

Thus the roots of $f(x) = 0$ are $1, -1, -3, \frac{-1}{3}, \frac{3+\sqrt{5}}{2}$,

$\frac{3-\sqrt{5}}{2}$.

Problem 9:

Solve $6x^6 - 5x^5 - 44x^4 + 44x^2 + 5x - 6 = 0$

<u>Soln</u>:

$$f(x) = 6x^6 - 5x^5 - 44x^4 + 44x^2 + 5x - 6 = 0$$

The given equation is a reciprocal equation of second type and of even degree.

$x^2 - 1$ is a factor of $f(x)$. By actual division.

1	6	-5	-44	0	44	5	-6	
	↓	6	1	-43	-43	1	6	
-1	6	1	-43	-43	1	6	0	
	↓	-6	5	38	5	-6		
	6	-5	-38	-5	6	0		

$$f(x) = (x^2 - 1)(6x^4 - 5x^3 - 38x^2 + 5x + 6).$$

$6x^4 - 5x^3 - 38x^2 + 5x + 6$ is a SRE

Dividing by x^2 and regrouping.

$$6\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) - 38 = 0$$

$x + \frac{1}{x} = y$ $\qquad$ $x^2 + \frac{1}{x^2} = y^2 - 2$

$$6y^2 - 12 - 5y - 38 = 0$$

$$6y^2 - 5y - 50 = 0$$

$$(3y - 10)(2y + 5) = 0$$

$$y = \frac{10}{3}, \frac{-5}{2}$$

$$y = \frac{10}{3} \Rightarrow x + \frac{1}{x} = \frac{10}{3}$$

$$3x^2 + 3 = 10x$$

$$3x^2 - 10x + 3 = 0$$

$$x = 3, \frac{1}{3}$$

$$y = \frac{-5}{2} \Rightarrow x + \frac{1}{x} = \frac{-5}{2}$$

$$2x^2 + 2 + 5x = 0$$

$$x = -2, \frac{-1}{2}$$

Thus the roots of $f(x) = 0$ are $1, -1, -2, \frac{-1}{2}, 3, \frac{1}{3}$

$$\left(\frac{z}{a_0}\right)^3 + \frac{3H}{a_0^2}\left(\frac{z}{a_0}\right) + \frac{G}{a_0^3} = 0$$

$$\text{(ie)} \quad z^3 + 3Hz + G = 0$$

Note. $z = a_0 y = a_0(x - h) = a_0x - a_0h = a_0x - a_0\left(-\frac{a_1}{a_0}\right) = a_0x + a_1$

Exercises.

1. Remove the second term from the equation $x^3 - 6x^2 + 10x - 3 \ldots$

2. Transform the equation $x^4 + 8x^3 + x - 5 = 0$ into one in which the second term is missing.

3. Solve the equation $x^3 + 6x^2 + 12x - 19 = 0$ by removing the second term.

4. Transform the equation $x^4 + 20x^3 + 143x^2 + 430x + 462 = 0$ into one in which the second term is missing and hence solve the given equation.

5. Solve the following equations by removing the second term.
   (i) $x^3 - 12x^2 - 48x - 72 = 0$
   (ii) $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$
   (iii) $x^4 + 16x^3 + 72x^2 + 64x - 129 = 0$
   (iv) $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0$

6. Show that the same transformation removes second and third terms of the equation $x^3 + 3x^2 + 3x + 28 = 0$ and hence solve it.

5.6 MULTIPLE ROOTS.

Definition. A complex number α is said to be a ***root of multiplicity m*** for the equation $f(x) = 0$ if $f(x) = (x - \alpha)^m \varphi(x)$ where $\varphi(\alpha) \neq 0$

Remark 1. α is a root of multiplicity m for $f(x) = 0$ iff $(x - \alpha)^m \mid f(x)$ and $(x - \alpha)^{m+1}$ does not divide $f(x)$

Remark 2. A root of multiplicity 2 is called a ***double root*** and a root of multiplicity 3 is called a ***triple root***

Theorem 14. α is a root of multiplicity m for the equation $f(x) = 0$ iff α is a root of multiplicity $m-1$ for the equation $f'(x) = 0$ where $f'(x)$ is the derivative of $f(x)$.

Proof. Let α be a root of multiplicity m for $f(x) = 0$ so that $f(x) = (x-\alpha)^m \varphi(x)$ where $\varphi(\alpha) \neq 0$.

Now, $f'(x) = (x-\alpha)^{m}\varphi'(x) + m(x-\alpha)^{m-1}\varphi(x)$

$= (x-\alpha)^{m-1}[m\varphi(x) + (x-\alpha)\varphi'(x)]$

$= (x-\alpha)^{m-1}\psi(x)$ where $\psi(x) = m\varphi(x) + (x-\alpha)\varphi'(x)$

Further $\psi(\alpha) = m\varphi(\alpha) \neq 0$ (since $\varphi(\alpha) \neq 0$) and hence α is a root of multiplicity $m-1$ for $f'(x) = 0$.

Conversely, let α be a root of multiplicity $m-1$ for $f'(x) = 0$, so that $(x-\alpha)^{m-1} \mid f'(x)$ and $(x-\alpha)^m \nmid f'(x)$. Let k be the largest integer such that $(x-\alpha)^k \mid f(x)$. Then $(x-\alpha)^{k+1} \nmid f(x)$. By the above argument $(x-\alpha)^{k-1} \mid f'(x)$ and $(x-\alpha)^k \nmid f'(x)$ so that α is a root of multiplicity $k-1$ to $f'(x)$. Hence $k-1 = m-1$. Thus $k = m$ and α is a root of multiplicity m for $f(x) = 0$.

Remark. Let α be a root of multiplicity m for $f(x) = 0$. By successive application of the above theorem, α is a root of multiplicity $m-1$ for $f'(x) = 0$, multiplicity $m-2$ $f''(x) = 0$, ..., a simple root for $f^{(m-1)}(x) = 0$ and $f^{(m)}(\alpha) \neq 0$. Thus α is a root of multiplicity m for $f(x) = 0$ iff $f(\alpha) = f'(\alpha) = \ldots\ldots\ldots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$.

Working Rule for finding the multiple root of an equation $f(x) = 0$.

(1) Find $f'(x)$.

(2) Find the H. C. F of $f(x)$ and $f'(x)$.

(3) Find the roots of the H. C. F.

(4) If α is a root of multiplicity m for the H. C. F., then α is a root of multiplicity $m+1$ for $f(x)$.

Problem 1. Find the multiple root of $4x^3 - 12x^2 - 15x - 4 = 0$ and solve completely.

Solution. Let $f(x) = 4x^3 - 12x^2 - 15x - 4$

$\therefore \quad f'(x) = 12x^2 - 24x - 15$

$= 3(4x^2 - 8x - 5)$

We now find the H. C. F. of $f(x)$ and $f'(x)$ as described below.

$2x$	$4x^2 - 8x - 5$	$4x^3 - 12x^2 - 15x - 4$	x
	$4x^2 + 2x$	$4x^3 - 8x^2 - 5x$	
-5	$-10x - 5$	$-4x^2 - 10x - 4$	-1
	$-10x - 5$	$-4x^2 + 8x + 5$	
	0	$-18x - 9$	
		$+9 \quad 2x + 1$	

$\therefore$ $2x + 1$ is the H. C. F. of $f(x)$ and $f'(x)$.

$\therefore$ $(2x + 1)^2$ is a factor of $f(x)$.

$\therefore$ $-\frac{1}{2}, -\frac{1}{2}$ are the double roots of $f(x) = 0$.

Since the given equation is of degree 3 it has three roots.
Let the third root be α_3.

Product of the roots $\alpha_3 = \frac{4}{4} = 1$.

$\therefore$ $(-\frac{1}{2})(-\frac{1}{2})\alpha_3 = 1$. Hence $\alpha_3 = 4$.

$\therefore$ The roots are $-\frac{1}{2}, -\frac{1}{2}, 4$.

Problem 2. Solve $4x^3 - 12x^2 - 15x - 4 = 0$ given that it has a double root.

Solution. Let $f(x) = 4x^3 - 12x^2 - 15x - 4$. ....(1)

$\therefore \quad f'(x) = 12x^2 - 24x - 15 = 3(4x^2 - 8x - 5)$

$= 3(2x + 1)(2x - 5)$

The roots of $f'(x) = 0$ are $x = -\frac{1}{2}$ and $x = \frac{5}{2}$

Now $f(-\frac{1}{2}) = 4(-\frac{1}{2})^3 - 12(-\frac{1}{2})^2 - 15(-\frac{1}{2}) - 4$

$= 0.$

$\therefore$ $-\frac{1}{2}$ is a common root of $f(x) = 0$ and $f'(x) = 0$.

$\therefore$ $-\frac{1}{2}$ is a double root of $f(x) = 0$.

$\therefore$ $(x + \frac{1}{2})^2$ is a factor of $f(x)$.

(ie) $(2x + 1)^2$ is a factor of $f(x)$.

(ie) $4x^2 + 4x + 1$ is a factor of $f(x)$.

Also $f(x) = (4x^2 + 4x + 1)(x - 4)$ (verify).

$\therefore$ The roots of $f(x) = 0$ are $-\frac{1}{2}, -\frac{1}{2}, 4$.

oblem 3. Solve the equation $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$ given tha as multiple roots.

$x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$ ....(1).

lution. Let $f(x) = x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$ ....(1).

$\therefore$ $f'(x) = 4x^3 - 18x^2 + 24x - 10 = 2(2x^3 - 9x^2 + 12x - 5)$.

We find the H. C. F of $f(x)$ and $f'(x)$ as described below.

					1	−6	12	−10	3	
2	2	−9	12	−5	2					
	2	−4	2		2	−12	24	−20	6	1
−5	2	−5	10	−5	2	−9	12	−5		
		−5	10	−5		−3	12	−15	6	
			0			−2				
						6	−24	30	−12	3
						6	−27	36	−15	
							3	−6	3	
					+3		1	−2	1	

$\therefore$ $x^2 - 2x + 1$ is the H. C. F. of $f(x)$ and $f'(x)$.

(ie) $(x - 1)^2$ is the H. C. F. of $f(x)$ and $f'(x)$.

$\therefore \quad (x-1)^3$ is a factor of $f(x)$.

Let α be the fourth root of the equation. Now the product of roots $S_4 = 3$. We get $\alpha = 3$.

The roots of $f(x) = 0$ are 1, 1, 1 and 3.

Problem 4 Find the multiple roots of $x^5 - x^4 + 2x^3 - 2x^2 + x - 1 = 0$ and hence solve.

Solution. Let $f(x) = x^5 - x^4 + 2x^3 - 2x^2 + x - 1 = 0$.

$$\therefore \quad f'(x) = 5x^4 - 4x^3 + 6x^2 - 4x + 1.$$

We find the H. C. F of $f(x)$ and $f'(x)$ as follows.

	5	−4	6	−4	1	1	−1	2	−2	1	−1
	2					5					
5	10	−8	12	−8	2	5	−5	10	−10	5	−5
	10	−15	10	−15		5	−4	6	−4	1	
		7	2	7	2		−1	4	−6	4	−5
		2					−5				
7		14	4	14	4		5	−20	30	−20	25
		14	−21	14	−21		5	−4	6	−4	1
			−25	0	−25			−16	24	−16	24
	÷ −25		1	0	1	÷ −8		2	−3	2	−3
									−3	0	−3
									−3	0	−3
										0	

$\therefore \quad x^2 + 1$ is the H. C.F. between $f(x)$ and $f'(x)$

$\therefore \quad (x^2 + 1)^2$ is a factor of $f(x)$.

(ie) $[(x+i)(x-i)]^2$ is a factor of $f(x)$.

$\therefore \quad i$ and $-i$ are the double roots of $f(x) = 0$.

Since $f(x) = 0$ is of degree five the fifth root α_5 can be found o

using the product of roots $S_5 = 1$.

$i \times i \times (-i) \times (-i) \times \alpha_5 = 1$ so that $\alpha_5 = 1$

The roots of $f(x) = 0$ are $i,\ i,\ -i,\ -i,\ 1$.

Problem 5. Prove that the equation $x^3 - 3qx + 2r = 0$ has a double root if $q^3 = r^2$.

Solution. Let $f(x) = x^3 - 3qx + 2r = 0$.

$$f'(x) = 3x^2 - 3q = 3(x^2 - q) = 0.$$

Let α be the double root of $f(x) = 0$.

Hence $f(\alpha) = 0$ and $f'(\alpha) = 0$.

$$\therefore \quad \alpha^3 - 3q\alpha + 2r = 0 \quad \ldots\ldots\ldots\ldots (1)$$

$$\alpha^2 - q = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Multiplying (2) by α and subtracting from (1) we get

$$-2q\alpha + 2r = 0$$

$$\therefore \quad \alpha = \frac{r}{q}.$$

Substituting in (2) we get $\dfrac{r^2}{q^2} - q = 0$

$$\therefore \quad r^2 = q^3.$$

Aliter. Let $f(x) = x^3 - 3qx + 2r = 0$.

Let α be a double root of $f(x) = 0$.

Let the roots of $f(x) = 0$ be $\alpha,\ \alpha,\ \beta$.

$$S_1 = \alpha + \alpha + \beta = 0 \Rightarrow \beta = -2\alpha \quad \ldots\ldots\ldots\ldots (1)$$

Now, $S_2 = \alpha^2 + 2\alpha\beta = -3q \quad \ldots\ldots\ldots\ldots (2)$

$$S_3 = \alpha^2\beta = -2r \quad \ldots\ldots\ldots\ldots (3)$$

Using (1) in (3) we get $-2\alpha^3 = -2r$. Hence $\alpha^3 = r$ $\ldots\ldots\ldots\ldots$ (4)

Using (1) in (2) we get $-3\alpha^2 = -3q$. Hence $\alpha^2 = q$ $\ldots\ldots\ldots\ldots$ (5)

$i \times i \times (-i) \times (-i) \times \alpha_5 = 1$ so that $\alpha_5 = 1$

The roots of $f(x) = 0$ are $i, i, -i, -i, 1$.

Problem 5. Prove that the equation $x^3 - 3qx + 2r = 0$ has a double root if $q^3 =$

Solution. Let $f(x) = x^3 - 3qx + 2r = 0$.

$$f'(x) = 3x^2 - 3q = 3(x^2 - q) = 0.$$

Let α be the double root of $f(x) = 0$.

Hence $f(\alpha) = 0$ and $f'(\alpha) = 0$.

$$\therefore \quad \alpha^3 - 3q\alpha + 2r = 0 \qquad \text{..................} \quad (1)$$

$$\alpha^2 - q = 0 \qquad \text{..................} \quad (2)$$

Multiplying (2) by α and subtracting from (1) we get

$$-2q\alpha + 2r = 0$$

$$\therefore \quad \alpha = \frac{r}{q}.$$

Substituting in (2) we get $\dfrac{r^2}{q^2} - q = 0$

$$\therefore \quad r^2 = q^3.$$

Aliter. Let $f(x) = x^3 - 3qx + 2r = 0$.

Let α be a double root of $f(x) = 0$.

Let the roots of $f(x) = 0$ be α, α, β.

$$S_1 = \alpha + \alpha + \beta = 0 \Rightarrow \beta = -2\alpha \qquad \text{..................} (1)$$

$$\text{Now, } S_2 = \alpha^2 + 2\alpha\beta = -3q \qquad \text{..................} (2)$$

$$S_3 = \alpha^2\beta = -2r \qquad \text{..................} (3)$$

Using (1) in (3) we get $-2\alpha^3 = -2r$. Hence $\alpha^3 = r$ ....(4)

Using (1) in (2) we get $-3\alpha^2 = -3q$. Hence $\alpha^2 = q$ ....(5)

From (4) and (5) we get $r^2 = q^3$.

Problem 6. Show that $x^5 - 15x^3 + 10x^2 + 60x - 72 = 0$ has thre equal and solve it completely.

Solution. Let $f(x) = x^5 - 15x^3 + 10x^2 + 60x - 72 = 0$

$$f'(x) = 5x^4 - 45x^2 + 20x + 60.$$

$$f''(x) = 20x^3 - 90x + 20.$$

$$= (x - 2)(2x^2 + 4x - 1).$$

$$f'''(x) = 60x^2 - 90.$$

Here $f''(2) = 0$. Also we notice that $f(2) = f'(2) = 0$ and $f'''(2) \neq$ so that $\alpha = 2$ is a triple root of $f(x) = 0$.

Now, we divide $f(x)$ by $x - 2$ successively.

2	1	0	−15	10	60	−72	
		2	4	−22	−24	72	
2	1	2	−11	−12	36	0	
		2	8	−6	−36		
2	1	4	−3	−18	0		
		2	12	18			
	1	6	9	0			

$\therefore \quad f(x) = (x - 2)^3 (x^2 + 6x + 9)$

$= (x - 2)^3 (x + 3)^2$

$\therefore$ The roots of $f(x) = 0$ are $2, 2, 2, -3, -3$.

Exercises.

1. Solve the following equations given that they have multiple r

   (i) $x^3 - x^2 - 8x + 12 = 0$

   (ii) $4x^3 + 4x^2 - 7x + 2 = 0$

(v) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$

(vi) $x^4 - 9x^2 + 4x + 12 = 0$

(vii) $x^4 + 3x^3 - 7x^2 - 15x + 18 = 0$

(viii) $8x^4 + 4x^3 - 18x^2 + 11x - 2 = 0$

(ix) $x^5 - 15x^3 + 10x^2 + 60x - 72 = 0$

2. Solve $x^4 + 7x^3 + 17x^2 + 17x + 6 = 0$ given that it has a double root.

3. Solve $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$ given that it has two equal roots.

4. Show that $x^5 - 5x^3 + 5x^2 - 1 = 0$ has three roots equal and find roots.

5. Find the values of a for which the equation $ax^3 - 9x^2 + 12x - 5 = 0$ has two equal roots and solve completely.

6. If the equation $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots show that the equal root is $\dfrac{bc - ad}{2(ac - b^2)}$.

7. If the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has three equal roots show that each of them is equal to $\dfrac{6c - ab}{3a^2 - 8b}$.

8. If the equation $x^5 - 10a^3x^2 + b^4x + c^5 = 0$ has three equal roots show that $ab^4 - 9a^5 + c^5 = 0$.

9. If the equation $x^n - qx^{n-m} + r = 0$ has two equal roots show that

$$\left[\ldots(n-m)\right]^n = \left[\frac{r}{m}(n-m)\right]^m.$$

10. If the equation $x^4 + 4ax^3 + 6bx^2 + c = 0$ has three equal roots show that (i) $3a^2 = 4b$ (ii) $27a^4 + 16c = 0$.

(v) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$

(vi) $x^4 - 9x^2 + 4x + 12 = 0$

(vii) $x^4 + 3x^3 - 7x^2 - 15x + 18 = 0$

(viii) $8x^4 + 4x^3 - 18x^2 + 11x - 2 = 0$

(ix) $x^5 - 15x^3 + 10x^2 + 60x - 72 = 0$

2. Solve $x^4 + 7x^3 + 17x^2 + 17x + 6 = 0$ given that it has a double
e.

3. Solve $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$ given that it has two equal
ts.

4. Show that $x^5 - 5x^3 + 5x^2 - 1 = 0$ has three roots equal and find
roots.

5. Find the values of a for which the equation $ax^3 - 9x^2 + 12x - 5 = 0$
two equal roots and solve completely.

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8. If the equation $x^5 - 10a^3x^2 + b^4x + c^5 = 0$ has three equal
s show that $ab^4 - 9a^5 + c^5 = 0$.

9. If the equation $x^n - qx^{n-m} + r = 0$ has two equal roots show that

$$\ldots - m)\Big]^{r} = \left[\frac{r}{m}(n - m)\right]^{m}.$$

10. If the equation $x^4 + 4ax^3 + 6bx^2 + c = 0$ has three equal roots
e that (i) $3a^2 = 4b$ (ii) $27a^4 + 16c = 0$.

5.7 NATURE AND POSITION OF ROOTS.

In this section we discuss the nature and position of roots of a polynomial equation $f(x) = 0$, such as the number of real roots and the position of a real root which is roughly determined by two consecutive integers between which the root lies. *Descarte's rule* of sign gives the upper limit on the number of real roots and *Sturm's theorem* gives the exact number of real roots.

Consider the equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. With out loss of generality we assume that the leading coefficient is positive. Further we assume that $a_n \neq 0$ so that 0 is not a root. $a_i x^{n-i}$ is called a *term* of the polynomial; a_i is called the *coefficient* and $n-i$ is called the *exponent* of the term. If $a_i = 0$ then we say that the corresponding term is missing.

Definition. Consider the signs of the terms of a polynomial taken in order from left to right. If the sign of a term is opposite to that of the preceding term, we say that a *change of sign* occurs. If the sign of a term is the same as that of the preceeding term, we say that *continuation of sign* occurs.

Example.

Consider the polynomial $f(x) = x^7 - 7x^6 + 2x^4 + x^3 - 7x^2 + 11x - 5$. The signs of the terms in the polynomial are $+ \; - \; + \; + \; - \; + \; -$. The polynomial has 5 changes of signs and one continuation of sign.

We now state without proof Descarte's rule of signs which gives useful information about the number of real roots of $f(x) = 0$.

Descarte's rule of sign.

The number of positive roots of $f(x) = 0$ cannot be more than the number of changes of sign in $f(x)$ and the number of negative roots of $f(x) = 0$ cannot be more than the number of changes of sign in $f(-x)$.

Corollary. Let n be the degree of $f(x)$. Let α and β denote the number of changes of sign in $f(x)$ and $f(-x)$ respectively. Then the equation

$f(x) = 0$ has atleast $n - (\alpha + \beta)$ complex roots.

Proof. By Descarte's rule, the number of real roots $\leq \alpha + \beta$.

Hence the number of complex roots $\geq n - (\alpha + \beta)$.

Solved problems

Problem 1. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has atleast four imaginary roots.

Solution. Let $f(x) = x^7 - 3x^4 + 2x^3 - 1 = 0$.

The signs of the terms are $+ \; - \; + \; -$.

There are 3 changes of signs.

$\therefore$ By Descarte's rule of signs $f(x) = 0$ can have at most 3 positive roots.

Now consider $f(-x) = -x^7 - 3x^4 - 2x^3 - 1$.

Here the signs of the terms are $- \; - \; - \; -$.

There is no change of sign.

There is no change of sign.

Hence $f(x) = 0$ can have atmost 3 real roots (positive and negative).

(x) The maximum number of real roots is 3.

But the given equation is of degree 7 and hence it has seven roots (real or complex).

$\therefore$ $f(x) = 0$ has atleast 4 imaginary roots.

Problem 2. Prove that the equation $x^4 + 3x - 1 = 0$ has two real and two imaginary roots.

Solution. Let $f(x) = x^4 + 3x - 1 = 0$.

The signs of the terms are $+ \; + \; -$. There is only one change of sign.

By Decarte's rule $f(x) = 0$ has atmost one positive root.

Also $f(-x) = x^4 - 3x - 1$ and $f(-x)$ has one change of sign

$f(x) = 0$ has atmost one negative root.

Thus $f(x) = 0$ has atmost 2 real roots.

Since $f(x)$ is of degree 4 it has atmost 4 roots.

Hence $f(x) = 0$ has atleast 2 complex roots.

$f(x) = 0$ has atleast $n - (\alpha + \beta)$ complex roots.

Proof. By Descarte's rule, the number of real roots $\le \alpha + \beta$.

Hence the number of complex roots $\ge n - (\alpha + \beta)$.

Solved problems

Problem 1. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has atleast four imaginary roots.

Solution. Let $f(x) = x^7 - 3x^4 + 2x^3 - 1 = 0$.

The signs of the terms are $+ \ - \ + \ -$.

There are 3 changes of signs.

$\therefore$ By Descarte's rule of signs $f(x) = 0$ can have at most 3 positive roots.

Now consider $f(-x) = -x^7 - 3x^4 - 2x^3 - 1$.

Here the signs of the terms are $- \ - \ - \ -$.

There is no change of sigh.

Hence $f(x) = 0$ can have atmost 3 real roots (positive and negative).

$\therefore$ The maximum number of real roots is 3.

But the given equation is of degree 7 and hence it has seven roots (real or complex).

$\therefore$ $f(x) = 0$ has atleast 4 imaginary roots.

Problem 2. Prove that the equation $x^4 + 3x - 1 = 0$ has two real and two imaginary roots.

Solution. Let $f(x) = x^4 + 3x - 1 = 0$.

The signs of the terms are $+ \ + \ -$. There is only one change of sign.

By Decarte's rule $f(x) = 0$ has atmost one positive root.

Also $f(-x) = x^4 - 3x - 1$ and $f(-x)$ has one change of sign

$f(x) = 0$ has atmost one negative root.

Thus $f(x) = 0$ has atmost 2 real roots.

Since $f(x)$ is of degree 4 it has atmost 4 roots.

Hence $f(x) = 0$ has atleast 2 complex roots.

But complex roots, will occur in conjugate pairs.

$\therefore$ There are exactly two complex roots and 2 real roots for $f(x)$...

Problem 3. Show that the equation $x^4 - ax^3 - bx - c = 0$ has atleast two complex roots for all positive values of a, b and c.

Solution. Let $f(x) = x^4 - ax^3 - bx - c = 0$.

For all positive values of a, b, c there is only one change of sign.

$\therefore$ $f(x) = 0$ has atmost one positive real root.

Also $f(-x) = x^4 + ax^3 + bx - c = 0$ and $f(-x)$ has only one change of sign.

Hence $f(x) = 0$ has atmost one negative real root.

$\therefore$ $f(x) = 0$ are has atleast two complex roots for all positive values of a, b, c.

Problem 4. Show that the equation $x^n + 1 = 0$ has no real root when n is even. When n is odd -1 is the only real root and the remaining roots are imaginary.

Solution. Let $f(x) = x^n + 1$.

Case (i). n is even. Then $f(-x) = (-x)^n + 1 = x^n + 1$.

Hence $f(x) = 0$ and $f(-x) = 0$ both has no change of sign.

$\therefore$ $f(x) = 0$ has no real root.

$\therefore$ All the n roots are imaginary.

Case (ii). n is odd. In this case, $f(x) = 0$ has no change of sign.

Hence it has no positive real root.

Also $f(-x) = (-x)^n + 1 = -x^n + 1$, which has one change of sign.

$\therefore$ $f(-x)$ has atmost one positive real root.

$\therefore$ $f(x)$ has atmost one negative real root.

Further $f(-1) = 0$ when n is odd.

$\therefore$ The only real root of $f(x) = 0$ is -1.

The remaining roots are imaginary.

But complex roots, will occur in conjugate pairs.

$\therefore$ There are exactly two complex roots and 2 real roots for $f(x)$ ...

Problem 3. Show that the equation $x^4 - ax^3 - bx - c = 0$ has atleast ... complex roots for all positive values of a, b and c.

Solution. Let $f(x) = x^4 - ax^3 - bx - c = 0$.

For all positive values of a, b, c there is only one change of sign.

$\therefore$ $f(x) = 0$ has atmost one positive real root.

Also $f(-x) = x^4 + ax^3 + bx - c = 0$ and $f(-x)$ has only one change of sign.

Hence $f(x) = 0$ has atmost one negative real root.

$\therefore$ $f(x) = 0$ are has atleast two complex roots for all positive values of a, b, c.

Problem 4. Show that the equation $x^n + 1 = 0$ has no real root when n is even. When n is odd -1 is the only real root and the remaining roots are imaginary.

Solution. Let $f(x) = x^n + 1$.

Case (i). n is even. Then $f(-x) = (-x)^n + 1 = x^n + 1$.

Hence $f(x) = 0$ and $f(-x) = 0$ both has no change of sign.

$\therefore$ $f(x) = 0$ has no real root.

$\therefore$ All the n roots are imaginary.

Case (ii). n is odd. In this case, $f(x) = 0$ has no change of sign.

Hence it has no positive real root.

Also $f(-x) = (-x)^n + 1 = -x^n + 1$, which has one change of sign.

$\therefore$ $f(-x)$ has atmost one positive real root.

$\therefore$ $f(x)$ has atmost one negative real root.

Further $f(-1) = 0$ when n is odd.

$\therefore$ The only real root of $f(x) = 0$ is -1.

The remaining roots are imaginary.

Rolles theorem.

Between two consecutive reall roots a and b of the equation $f(x) = 0$, the derivative $f'(x) = 0$ has atleast one real root.

We get the following results which are immediate from the theorem.

Result 1. If all the roots of $f(x) = 0$ are real then all the roots of $f'(x) = 0$ are also real.

Result 2. If $f'(x) = 0$ has r real roots then $f(x) = 0$ can not have more than $r + 1$ real roots.

Note. The sign $f(x)$ at ∞ and $-\infty$ is determined by the following

(i) $f(\infty)$ is positive when the leading coefficient is positive.

(ii) $f(\infty)$ is negative when the leading coefficient is negative.

(iii) $f(-\infty)$ is negative when the leading coefficient is positive and the degree of $f(x)$ is odd.

(iv) $f(-\infty)$ is positive when the leading coefficient is positive and the degree of $f(x)$ is even.

Solved problems.

Problem 1. Find the nature of roots of $2x^3 - 9x^2 + 12x + 3 = 0$

Solution. Let $f(x) = 2x^3 - 9x^2 + 12x + 3$.

$$\therefore \quad f'(x) = 6x^2 - 18x + 12 = 0.$$
$$= 6(x^2 - 3x + 2)$$
$$= 6(x - 1)(x - 2).$$

$\therefore$ The roots of $f'(x) = 0$ are 1 and 2 which are real.

$\therefore$ The roots of $f(x) = 0$ will lie in the interval $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. Now we have the following table of signs at these points of the interval.

x	$-\infty$	1	2	∞
$f(x)$	$-$	$+$	$+$	$+$

Here $f(x) = 0$ has one change of sign.

$\therefore$ $f(x) = 0$ has one real root α (say) in the interval $(-\infty , 1)$.

Now, $f(0) = 3$ and $f(1) = 8$. There is no change of sign for $f(x) = 0$ between 0 and 1.

$\therefore$ $\alpha \notin (0 , 1)$. Hence $\alpha \in (-\infty , 0)$.

$\therefore$ α is a negative real root of $f(x) = 0$. Since the given equation is of degree three the other two roots must be imaginary roots.

Hence $f(x) = 0$ has one negative root and two imaginary roots.

Problem 2. Find the nature of the roots of $x^4 + 4x^3 - 20x^2 + 10 = 0$

Solution. Let $f(x) = x^4 + 4x^3 - 20x^2 + 10$.

$$\therefore \quad f'(x) = 4x^3 + 12x^2 - 40x$$
$$= 4x(x^2 + 3x - 10)$$
$$= 4x(x + 5)(x - 2).$$

$f'(x) = 0 \Rightarrow x = -5 , 0 , 2$

$f'(x) = 0$ has 3 real roots.

$\therefore$ $f(x) = 0$ can not have more than 4 real roots (by result 2).

$\therefore$ The roots of $f(x) = 0$ will lie in the intervals $(-\infty , -5)$, $(-5 , 0)$, $(0 , 2)$, $(2 , \infty)$. We have the following table giving the signs of $f(x)$ at these points

x	$-\infty$	-5	0	2	∞
$f(x)$	+	−	+	−	+

Here, $f(x) = 0$ has 4 changes of signs.

$\therefore$ $f(x)$ has four real roots. Since the given equation is of degree 4 all the roots of $f(x) = 0$ are real which lie in the four intervals $(-\infty , -5)$, $(-5 , 0)$, $(0 , 2)$, $(2 , \infty)$. Two of them are negative and other two positive.

Problem 3. Find the range of k for which $x^4 - 14x^2 + 24x - k = 0$ has all real roots.

Solution. Let $f(x) = x^4 - 14x^2 + 24x - k$.

Here $f(x) = 0$ has one change of sign.

$\therefore$ $f(x) = 0$ has one real root α (say) in the interval $(-\infty, 1)$.

Now, $f(0) = 3$ and $f(1) = 8$. There is no change of sign for $f(x) = 0$ between 0 and 1.

$\therefore$ $\alpha \notin (0, 1)$. Hence $\alpha \in (-\infty, 0)$.

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x	$-\infty$	-5	0	2	∞
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Here, $f(x) = 0$ has 4 changes of signs.

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Problem 3. Find the range of k for which $x^4 - 14x^2 + 24x - k = 0$ has all real roots.

Solution. Let $f(x) = x^4 - 14x^2 + 24x - k$.

$\therefore \quad f'(x) = 4x^3 - 28x + 24$

$\quad\quad\quad = 4(x^3 - 7x + 6)$

$\quad\quad\quad = 4(x - 1)(x^2 + x - 6)$

$\quad\quad\quad = 4(x - 1)(x + 3)(x - 2)$

$f'(x) = 0 \Rightarrow x = -3, 1, 2$ (real).

$\therefore$ The roots of $f(x) = 0$ will lie in the interval $(-\infty, -3)$, $(-3, 1)$, $(1, 2)$.

Now, we have the table giving the signs of $f(x)$ at these points.

x	$-\infty$	-3	1	2	∞
$f(x)$	$+$	$-117-k$	$11-k$	$8-k$	$+$

Given $f(x) = 0$ has all real roots. This is possible only if we have the following inequalities.

$-117 - k < 0$ ....(1)

$11 - k > 0$ ....(2)

$8 - k < 0$ ....(3)

$$\left.\begin{array}{l} (1) \Rightarrow 117 + k \geq 0 \Rightarrow k \geq -117 \\ (2) \Rightarrow \quad\quad\quad\quad\quad k < 11 \\ (3) \Rightarrow \quad\quad\quad\quad\quad k > 8 \end{array}\right] \rightarrow 8 < k < 1$$

$\therefore$ If $k \in (8, 11)$ then $f(x) = 0$ has all roots rea

Exercises.

1. Find the nature of roots of the following equations,
   (i) $4x^3 - 21x^2 + 18x + 20 = 0$
   (ii) $4x^3 - 21x^2 + 18x + 30 = 0$
   (iii) $3x^4 - 8x^3 - 6x^2 + 24x - 7 = 0$.
2. Find the range of values of a for which the following equation have all real and distinct roots.
   (i) $x^3 - 3x + a = 0$

$(2)^4 - 14(2)^2 + 24(2) - k = 0$

$16 - 56 + 48 - k = 0$ $8 - k$

$64 - 56 - k = 0$

$$\therefore \quad f'(x) = 4x^3 - 28x + 24$$
$$= 4(x^3 - 7x + 6)$$
$$= 4(x - 1)(x^2 + x - 6)$$
$$= 4(x - 1)(x + 3)(x - 2)$$
$$f'(x) = 0 \Rightarrow x = -3,\ 1,\ 2 \quad (\text{real}).$$

$\therefore$ The roots of $f(x) = 0$ will lie in the interval $(-\infty, -3)$, $(-3, 1)$, $(1, 2)$.

Now, we have the table giving the signs of $f(x)$ at these points.

x	$-\infty$	-3	1	2	∞
$f(x)$	+	$-117-k$	$11-k$	$8-k$	+

Given $f(x) = 0$ has all real roots. This is possible only if we have the following inequalities.

$$-117 - k < 0 \qquad \text{.................(1)}$$
$$11 - k > 0 \qquad \text{.................(2)}$$
$$8 - k < 0 \qquad \text{.................(3)}$$

$$\left.\begin{array}{ll} (1) \Rightarrow 117 + k \geq 0 \Rightarrow & k \geq -117 \\ (2) \Rightarrow & k < 11 \\ (3) \Rightarrow & k > 8 \end{array}\right] \Rightarrow 8 < k < 11$$

$\therefore$ If $k \in (8, 11)$ then $f(x) = 0$ has all roots real.

Exercises.

1. Find the nature of roots of the following equations.

   (i) $4x^3 - 21x^2 + 18x + 20 = 0$

   (ii) $4x^3 - 21x^2 + 18x + 30 = 0$

   (iii) $3x^4 - 8x^3 - 6x^2 + 24x - 7 = 0.$

2. Find the range of values of a for which the following equations have all real and distinct roots.

   (i) $x^3 - 3x + a = 0$

(v) $x^4 + 4x^3 - \ldots$

(vii) $3x^4 + 8x^3 - 6x^2 - 24x + k = 0$.

$+ 24(-3) - k = 0$

$(-3)^4 - 14(-3)^2 \quad - k = 0$

Sturm's Theorem.

$81 - 14(9) - 72$

Sturm's function.

Let $f(x)$ be a given polynomial. Let $f_1(x)$ be its *derivative*. We perform the operation of finding the H. C. F. of $f(x)$ and $f_1(x)$ with the following modification.

" *The sign of each remainder is to be changed before it is used as divisor* ".

Let the modified remainders be $f_2(x)$, $f_3(x)$, ... , $f_r(x)$. Then the functions $f(x)$, $f_1(x)$, $f_2(x)$, ... are called the *Sturm's functions.*

Hence the Sturm's functions are connected by the following equations.

126 72 14 9 198 126 $71 - 126 - 72 -$

$$f(x) = q_1 f_1(x) - f_2(x)$$
$$f_1(x) = q_2 f_2(x) - f_3(x)$$
$$f_2(x) = q_3 f_3(x) - f_4(x)$$
....

....

$$f_{r-2}(x) = q_{r-1} f_{r-1}(x) - f_r(x)$$ where q_1 , q_2 , ... , q_{r-1} are the quotients in this process described above.

We now state without proof, the following theorem which gives the exact number of real roots of the equation $f(x) = 0$.

Theorem. (**Sturm's theorem**)

Let $f(x)$ be a polynomial and a , b be any two real numbers such that $a < b$. Then the number of distinct real roots of $f(x) = 0$ which lie between a and b (*any multiple root, which may exist, being counted once only*) is equal to the difference between the number of changes of signs in the sequence of Sturm's functions $f(x)$, $f_1(x)$, $f_2(x)$, ... when $x = a$ and the number of changes of signs in the above sequence when $x = b$.

Remark 1. If $f(x) = 0$ has no multiple roots then $f(x)$ and $f'(x)(= f_1(x))$ have no common factor (refer theorem). Hence the last of the Sturm's functions is a constant polynomial.

If $f(x) = 0$ has multiple roots then the last of the Sturm's functions is the H. C. F. of $f(x)$ and $f_1(x)$.

Remark 2. In the process of calculating Sturm's functions if at any stage we arrive at a function $f_r(x)$ such that all the roots of $f_r(x) = 0$ are imaginary then the process need not be continued and we can use $f(x)$, $f_1(x)$, ..., $f_r(x)$ instead of the complete sequence of Sturm's functions.

Remark 3. For any Sturm's function say $f_m(x)$, if $f_m(a) = 0$ then in counting the number of changes of signs in the sequence $f(a)$, $f_1(a)$, ..., $f_r(a)$, we may take the sign of $f_m(a)$ as either positive or negative.

Working rule for finding the number of real roots of $f(x) = 0$.

1. *Sturm's function.*

Calculate the Sturm's functions for $f(x)$ by applying the method of finding H. C. F. of $f(x)$ and $f_1(x)(= f'(x))$.

Let the sequence of Sturm's functions be $f(x)$, $f_1(x)$, $f_2(x)$, ..., $f_r(x)$.

2. *Negative roots.*

To find the number of negative real roots apply Sturm's theorem by taking $a = -\infty$ and $b = 0$. For this, calculate the number of changes of signs in the above sequence when $x = -\infty$ and when $x = 0$. The difference gives the number of negative real roots.

3. *Positive roots.*

To find the number of positive real roots apply Sturm's theorem by taking $a = 0$ and $b = \infty$. For this, calculate the number of changes of signs in the sequence when $x = 0$ and when $x = \infty$. The difference gives the number of positive real roots.

4. *Position of roots.* If we want to find the interval in which the roots lie then put $x = 1, 2, 3,$ .... for positive roots and $x = -1, -2, -3,$ .... for negative roots and the intervals can be determined.

$$f(x) = x^4 - 2x^3 - 3x^2 + 10x - 4.$$
$$f'(x) = 4x^3 - 6x^2 - 6x + 10.$$
$$f_2(x) = 9x^2 - 27x + 11$$
$$f_3(x) = - 8x - 3 \text{ and } f_4(x) = - 1433$$

Problem 2. Find the number of real roots of $x^4 + 4x^3 - 4x - 13 = 0$ by Sturm's theorem.

Solution. Let $f(x) = x^4 + 4x^3 - 4x - 13.$

$$\therefore \quad f'(x) = 4x^3 + 12x^2 - 4 = 4(x^3 + 3x^2 - 1)$$

We find, first of all, the Sturm's functions for $f(x)$ as follows by the method of H. C. F.

x	$x^3 + 3x^2 - 1$	$x^4 + 4x^3 - 4x - 13$	x
	$x^3 + x^2 + 4x$	$x^4 + 3x^3 - x$	
2	$2x^2 - 4x - 1$	$x^3 - 3x - 13$	1
	$2x^2 + 2x + 8$	$x^3 + 3x^2 - 1$	
	$-6x - 9$	$-3x^2 - 3x - 12$	
	$+3 \quad -2x - 3$	$+3 \quad -x^2 - x - 4$	
	$\therefore f_3(x) = 2x + 3$	$\therefore f_2(x) = x^2 + x + 4$	
		2	
		$2x^2 + 2x + 8$	x
		$2x^2 + 3x$	
		$-x + 8$	
		2	
		$-2x + 16$	-1
		$-2x - 3$	
		19	
		$\therefore f_4(x) = -19$	

$\therefore$ The Sturm's functions are

$$f(x) = x^4 + 4x^3 - 4x - 13$$

$f'(x) = 4(x^3 + 3x^2 - 1)$

$f_2(x) = x^2 + x + 4$

$f_3(x) = 2x + 3$

$f_4(x) = -19.$

The signs of Sturm's functions by substituting $-\infty, 0, \infty,$ for x are shown in the following table.

(i)	(ii)	(iii)	(iv)
x	$-\infty$	0	∞
$f(x)$	+	-	+
$f'(x)$	-	-	+
$f_2(x)$	+	+	+
$f_3(x)$	-	+	+
$f_4(x)$	-	-	-
	3	2	1

The difference of changes of signs when $x = -\infty$ and $x = \infty$ in Sturm's function is 2. [from column (ii) and column (iv)].

There are only 2 real roots.

The difference of changes of signs when $x = -\infty$ and $x = 0$ is 1 [from column (ii) and column (iii)].

Hence there is only one negative real root.

The difference of change of signs when $x = 0$ and $x = \infty$ is 1 [from column (iii) and column (iv)].

There is only one positive real root.

Hence the given equation $f(x) = 0$ has two real roots one is positive and the other is negative.

Example 3. Find the number and posititon of the real roots of

$$x^3 - 7x + 7 = 0.$$

Solution. Let $f(x) = x^3 - 7x + 7 = 0.$

$\therefore \quad f'(x) = 3x^2 - 7.$

We find the Sturm's functions as follows.

$$f'(x) = 4(x^3 + 3x^2 - 1)$$
$$f_2(x) = x^2 + x + 4$$
$$f_3(x) = 2x + 3$$
$$f_4(x) = -19.$$

The signs of Sturm's functions by substituting $-\infty, 0, \infty$, for x are shown in the following table.

(i)	(ii)	(iii)	(iv)
x	$-\infty$	0	∞
$f(x)$	+	−	+
$f'(x)$	−	−	+
$f_2(x)$	+	+	+
$f_3(x)$	−	+	+
$f_4(x)$	−	−	−

3 2 1

The difference of changes of signs when $x = -\infty$ and $x = \infty$ in Sturm's function is 2. [from column (ii) and column (iv)].

There are only 2 real roots.

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The difference of change of signs when $x = 0$ and $x = \infty$ is 1 [from column (iii) and column (iv)].

There is only one positive real root.

Hence the given equation $f(x) = 0$ has two real roots one is positive and the other is negative.

Problem 3. Find the number and posititon of the real roots of

$$x^3 - 7x + 7 = 0.$$

Solution. Let $f(x) = x^3 - 7x + 7 = 0.$

$\therefore \quad f'(x) = 3x^2 - 7.$

$3x^2$

We find the Sturm's functions as follows.

	$3x^2-7$	x^3-7x+7	
	2	3	
$3x$	$6x^2-14$	$3x^3-21x+21$	x
	$6x^2-9x$	$3x^3-7x$	
	$9x-14$	$-14x+21$	
	2	$+7 \quad -2x+3$	
9	$18x-28$	$\therefore f_2(x)=2x-3$	
	$18x-27$		
	-1		
	$\therefore f_3(x)=1$		

$\therefore$ The Sturm's functions are $f(x)=x^3-7x+7$.

$$f'(x)=3x^2-7.$$
$$f_2(x)=2x-3$$
$$f_3(x)=1$$

We write the table of signs of Sturm's functions when x takes values $-\infty$, 0, ∞.

(i)	(ii)	(iii)	(iv)	(v)	(vi)		(vii)	(viii)	(ix)
x	$-\infty$	0	∞	1	2	....	-1	-2	-3
$f(x)$	$-$	+	+	+	+	....	+	+	+
$f'(x)$	+	$-$	+	$-$	+	....			
$f_2(x)$	$-$	$-$	+	$-$	+	....			
$f_3(x)$	+	+	+	+	+	....			
No. of changes of signs	3	2	0	2	0				

From columns (ii) and (iv) the difference between the number of changes of signs of Sturm's functions at $x=-\infty, x=\infty$ is 3. Hence there are 3 real roots for $f(x)=0$.

From columns (ii) and (iii) we observe the difference between the number of changes of signs of Sturm's functions at $x=-\infty, x=0$ is 1

$\therefore$ One real root of $f(x)=0$ is negative.

	$3x^2 - 7$ 2	$x^3 - 7x + 7$ 3	
$3x$	$6x^2 - 14$ $6x^2 - 9x$	$3x^3 - 21x + 21$ $3x^3 - 7x$	x
	$9x - 14$ 2	$-14x + 21$ $+7 \quad -2x + 3$	
9	$18x - 28$ $18x - 27$	$\therefore f_2(x) = 2x - 3$	
	-1 $\therefore f_3(x) = 1$		

$\therefore$ The Sturm's functions are $f(x) = x^3 - 7x + 7.$

$$f'(x) = 3x^2 - 7.$$
$$f_2(x) = 2x - 3$$
$$f_3(x) = 1$$

We write the table of signs of Sturm's functions when x takes $-\infty$, 0, ∞.

(i)	(ii)	(iii)	(iv)	(v)	(vi)		(vii)	(viii)	(ix)
x	$-\infty$	0	∞	1	2	....	-1	-2	-3
$f(x)$	$-$	+	+	+	+	....	+	+	+
$f'(x)$	+	$-$	+	$-$	+	....			
$f_2(x)$	$-$	$-$	+	$-$	+	....			
$f_3(x)$	+	+	+	+	+	....			
No. of changes of signs	3	2	0	2	0				

From columns (ii) and (iv) the difference between the number of changes of signs of Sturm's functions at $x = -\infty, x = \infty$ is 3. Hence there are 3 real roots for $f(x) = 0$.

From columns (ii) and (iii) we observe the difference between the number of changes of signs of Sturm's functions at $x = -\infty, x = 0$ is 1

$\therefore$ One real root of $f(x) = 0$ is negative.

$3x^2 - 7$ 2	$x^3 - 7x + 7$ 3
$3x$ $6x^2 - 14$ $6x^2 - 9x$	$3x^3 - 21x + 21$ $3x^3 - 7x$ x
$9x - 14$ 2	$-14x + 21$ $+7$ $-2x + 3$
9 $18x - 28$ $18x - 27$	$\therefore f_2(x) = 2x - 3$
-1 $\therefore f_3(x) = 1$	

$\therefore$ The Sturm's functions are $f(x) = x^3 - 7x + 7$.

$$f'(x) = 3x^2 - 7.$$
$$f_2(x) = 2x - 3$$
$$f_3(x) = 1$$

We write the table of signs of Sturm's functions when $x = -\infty, 0, \infty$.

(i)	(ii)	(iii)	(iv)	(v)	(vi)		(vii)	(viii)	(ix)
x	$-\infty$	0	∞	1	2	....	-1	-2	-3
$f(x)$	$-$	+	+	+	+	....	+	+	+
$f'(x)$	+	$-$	+	$-$	+	....			
$f_2(x)$	$-$	$-$	+	$-$	+	....			
$f_3(x)$	+	+	+	+	+	....			
No. of changes of signs	3	2	0	2	0				

From columns (ii) and (iv) the difference between the number of changes of signs of Sturm's functions at $x = -\infty, x = \infty$ is 3. Hence there are 3 real roots for $f(x) = 0$.

From columns (ii) and (iii) we observe the difference between the number of changes of signs of Sturm's functions at $x = -\infty, x = 0$ is 1

$\therefore$ One real root of $f(x) = 0$ is negative.

... columns (iii) and (iv) there are two positive real roots for ...

... now find the position of the roots for which we have written the ... (v) to (x).

... columns (v) and (vi) we observe that the two positive real roots lie ... 1 and 2

... columns (ix) and (x) there is a change of sign. Hence the negative ... lies between -3 and -4.

... the given equation has all three roots real of which two are positive ... the interval (1, 2) and the negative root lying in the interval ... 1).

... 4 Find the nature and position of the roots of the equation ... $^4 + x^3 - x^2 - 2x - 1 = 0$ by Sturm's functions.

$5x^4 + 8x^3 + 3x^2 - 2x - 2$	$x^5 + 2x^4 + x^3 - x^2 - 2x - 1$	
2	5	
$10x^4 + 16x^3 + 6x^2 - 4x - 4$	$5x^5 + 10x^4 + 5x^3 - 5x^2 - 10x - 5$	x
$10x^4 + 35x^3 + 60x^2 + 35x$	$5x^5 + 8x^4 + 3x^3 - 2x^2 \quad 2x$	
$-19x^3 - 54x^2 - 39x - 4$	$2x^4 + 2x^3 - 3x^2 - 8x - 5$	
2	5	
$-38x^3 - 108x^2 - 78x - 8$	$10x^4 + 10x^3 - 15x^2 \quad 40x \quad 25$	2
$-38x^3 - 133x^2 - 228x - 133$	$10x^4 + 16x^3 + 6x^2 - 4x - 4$	
$25x^2 + 150x + 125$	$-6x^3 - 21x^2 - 36x - 21$	
$x^2 + 6x + 5$	$-2x^3 - 7x^2 - 12x - 7$	
$f_3(x) = -x^2 - 6x - 5$	$\therefore f_2(x) = 2x^3 + 7x^2 + 12x + 7$	
$-x^2 - x$	$2x^3 + 12x^2 + 10x$	$-2x$
$-5x - 5$	$-5x^2 + 2x + 7$	5
$-5x - 5$	$-5x^2 - 30x - 25$	
0	$32x + 32$	
$f_5(x) = 0$	$+32 \qquad x + 1$	
	$f_4(x) = -x - 1.$	

Let $f(x) = x^5 + 2x^4 + x^3 - x^2 - 2x - 1$.

$\therefore \quad f'(x) = 5x^4 + 8x^3 + 3x^2 - 2x - 2$

We get the Sturm's functions as follows.

$\therefore$ Sturm's functions are $f(x) = x^5 + 2x^4 + x^3 - x^2 - 2x - 1$

$f'(x) = 5x^4 + 8x^3 + 3x^2 - 2x - 2$

$f_2(x) = 2x^3 + 7x^2 - 12x - 7$

$f_3(x) = -x^2 - 6x - 5$

$f_4(x) = -x - 1$ and $f_5(x) = 0$

We know, from the definition of Sturm's functions that $f_3(x) = q_4 f_4(x) - f_5(x)$. Since $f_5(x) = 0$, $f_4(x)$ divides $f_3(x)$. Hence $f_4(x) = -x - 1$ is the H. C. F. of $f(x)$ and $f'(x)$

(ie) $-x-1$ is a factor of $f(x)$ and $f'(x)$.

$\therefore$ By theorem14, $(-x-1)^2$ is a factor of $f(x)$.

(ie) $(x+1)^2$ is a factor of $f(x)$.

$\therefore$ -1 is a double root of $f(x) = 0$.

Now we write the table of signs of Sturm's functions

x	$-\infty$	0	∞	1	-1
$f(x)$	$-$	$-$	$+$	0	0
$f'(x)$	$+$	$-$	$+$		
$f_2(x)$	$-$	$+$	$+$		
$f_3(x)$	$-$	$-$	$-$		
$f_4(x)$	$+$	$-$	$-$		
No of change s of signs	3	2	1		

From the above table we observe that there are only two real roots of $f(x) = 0$ (from columns (ii) and (iv)) of which one is positive (from columns (iii) and (iv)) and the other is negative (from columns (ii) and (iii)) [By Sturm's theorem for multiple root -1 is counted once though it is a multiple root].

Further we note that $f(1) = 0$ and $f(-1) = 0$.

Hence $x = -1$ and $x = 1$ are the roots.

Hence $-1, -1, 1$ are the real roots of $f(x) = 0$ and the remaining two roots are complex.

Let $f(x) = x^5 + 2x^4 + x^3 - x^2 - 2x - 1$.

$\therefore \quad f'(x) = 5x^4 + 8x^3 + 3x^2 - 2x - 2$

We get the Sturm's functions as follows.

$\therefore$ Sturm's functions are $f(x) = x^5 + 2x^4 + x^3 - x^2 - 2x - 1$

$$f'(x) = 5x^4 + 8x^3 + 3x^2 - 2x - 1$$

$$f_2(x) = 2x^3 + 7x^2 - 12x - 7$$

$$f_3(x) = -x^2 - 6x - 5$$

$$f_4(x) = -x - 1 \text{ and } f_5(x) = 0.$$

We know, from the definition of Sturm's functions that $f_3(x) = q_4 f_4(x) - f_5(x)$. Since $f_5(x) = 0$, $f_4(x)$ divides $f_3(x)$.

Hence $f_4(x) = -x - 1$ is the H. C. F. of $f(x)$ and $f'(x)$

(ie) $-x-1$ is a factor of $f(x)$ and $f'(x)$.

$\therefore$ By theorem14, $(-x-1)^2$ is a factor of $f(x)$.

(ie) $(x+1)^2$ is a factor of $f(x)$.

$\therefore \quad -1$ is a double root of $f(x) = 0$.

Now we write the table of signs of Sturm's functions

x	$-\infty$	0	∞	1	-1
$f(x)$	$-$	$-$	$+$	0	0
$f'(x)$	$+$	$-$	$+$		
$f_2(x)$	$-$	$+$	$+$		
$f_3(x)$	$-$	$-$	$-$		
$f_4(x)$	$+$	$-$	$-$		
No of change s of signs	3	2	1		

From the above table we observe that there are only two re[illegible] $f(x) = 0$ (from columns (ii) and (iv)) of which one is positive (from [illegible] (iii) and (iv)) and the other is negative (from columns (i) a[illegible] [By Sturm's theorem for multiple root -1 is counted once though i[illegible] multiple root].

Further we note that $f(1) = 0$ and $f(-1) = 0$.

Hence $x = -1$ and $x = 1$ are the roots.

Hence $-1, -1, 1$ are the real roots of $f(x) = 0$ and the remaining [illegible] roots are complex.

Example 4. Find the number and position of the real root of $x^6 - 2x^2 + 3x - 4 = 0$.

Let $f(x) = x^6 - 2x^2 + 3x - 4 = 0$.

$f'(x) = 6x^5 - 4x + 3$.

We now find the Sturm's functions.

$$
\begin{array}{r|l|l}
6x^5 - 4x + 3 & x^6 - 2x^2 + 3x - 4 & \\
 & 6 & \\
\hline
 & 6x^6 - 12x^2 + 18x - 24 & x \\
 & 6x^6 - 4x^2 + 3x & \\
\hline
 & \quad - 8x^2 + 15x - 24 & \\
 & \therefore f_2(x) = 8x^2 - 15x + 24 &
\end{array}
$$

We notice that all roots of $f_2(x) = 0$ are imaginary, since its discriminant $(-15)^2 - 4\times 8\times 24$ is $-$ ive.

Hence we need not calculate Sturm's functions beyond $f_2(x)$.

Now the table of signs of Sturm's functions is

x	$-\infty$	0	∞	1	2	-1	-2
$f(x)$	+	−	+	−	+	−	+
$f_1(x)$	−	+	+				
$f_2(x)$	+	+	+				
No. of changes of sign	2	1	0				

From the table we observe that there are two real roots for $f(x) = 0$ of which one is negative and one is positive. (verify)

The positive root lies between 1 and 2 (since $f(1) = -$ ive and $f(2) = +$ ive) and the negative root lies between -1 and -2 (since $f(-1) = -$ ive and $f(-2) = +$ ive).

Example 4. Find the number and position of the real root of $x^6 - 2x^2 + 3x - 4 = 0$.

Solution. Let $f(x) = x^6 - 2x^2 + 3x - 4 = 0$.

$f'(x) = 6x^5 - 4x + 3$.

We now find the Sturm's functions.

$$6x^5 - 4x + 3 \;\Big|\; \frac{x^6 - 2x^2 + 3x - 4}{6}$$

$$6x^6 - 12x^2 + 18x - 24 \qquad x$$

$$6x^6 - 4x^2 + 3x$$

$$-8x^2 + 15x - 24$$

$$\therefore f_2(x) = 8x^2 - 15x + 24$$

We notice that all roots of $f_2(x) = 0$ are imaginary, since its discriminant $(-15)^2 - 4 \times 8 \times 24$ is − ive.

Hence we need not calculate Sturm's functions beyond $f_2(x)$.

Now the table of signs of Sturm's functions is

$-\infty$	0	∞	1	2	−1	−2
+	−	+	−	+	−	+
−	+	+				
+	+	+				
2	1	0				

From the table we observe that there are two real roots for $f(x) = 0$ of which one is negative and one is positive. (verify)

The positive root lies between 1 and 2 (since $f(1)$ = − ive and $f(2)$ = + ive) and the negative root lies between −1 and −2 (since $f(-1)$ = − ive and $f(-2)$ = + ive).

1. **Newton's method.**

Suppose that the equation $f(x) = 0$ has a single unrepeated root in an interval (α, β) where $\beta - \alpha$ is small. Hence $f'(x) \neq 0$ for all $x \in (\alpha, \beta)$.

Now, let $\alpha_1 = \alpha + h$ be the exact value of the root. $f(\alpha + h) = 0$. Hence by Taylor's expansion we have

$$\therefore \quad f(\alpha) + \frac{h}{1!}f'(\alpha) + \frac{h^2}{2!}f''(\alpha) + \cdots = 0.$$

Omitting h^2 and higher powers of h we get $f(\alpha) + hf'(\alpha) = 0$

$$\therefore \quad h = -\frac{f(\alpha)}{f'(\alpha)}.$$

$$\therefore \quad \alpha_1 = \alpha - \frac{f(\alpha)}{f'(\alpha)}$$ is an approximate value of the root.

By repeating this process the approximation can be carried out to any required degree of accuracy.

Solved problems

Problem 1. Show that $x^3 + 3x - 1 = 0$ has only one real root and calculate it correct to two places of decimals.

Solution. Since there is only one change of sign there cannot be more than one positive root. Further changing x into $-x$ the equation becomes $-x^3 - 3x - 1 = 0$. (ie) $x^3 + 3x + 1 = 0$.

This equation has no change of sign. Hence it has no negative root.

Since the imaginay roots occur in conjugate pairs the given equation has two imaginary roots and only one real root.

Now, we find the real root by Newton's method approximately to two places of decimals.

Let $f(x) = x^3 + 3x - 1$.

Since $f(0) = -1 < 0$ and $f(1) = 3 > 0$, the root lies between 0 and 1 (first approximation).

Let $0 + h$ be the actual value of the root.

$$\therefore \quad h = -\frac{f(0)}{f'(0)}$$

1. Newton's method.

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Problem 1. Show that $x^3 + 3x - 1 = 0$ has only one real root and calculate it correct to two places of decimals.

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Let $0 + h$ be the actual value of the root.

$$\therefore \quad h = -\frac{f(0)}{f'(0)}$$

Now, $f'(x) = 3x^2 + 3.$

$\therefore$ $f'(0) = 3.$

$\therefore$ $h = \frac{1}{3} = 0.3.$

Hence the root is $0 + 0.3 = 0.3$ (second approximation)

For third approximation, let the root be

$0.3 + h_1$ where $h_1 = -\frac{f(0.3)}{f'(0.3)}$.

$$\therefore \quad h_1 = \frac{(0.3)^3 + 3(0.3) - 1}{3(0.3)^2 + 3}.$$

$= 0.022.$

$\therefore$ The root is $= 0.3 + 0.022..$

$= 0.32$ (up to two places of decimals).

Problem 2. Find correct to 2 places of decimals the root of the equation $x^4 - 3x + 1$ that lies between 1 and 2 .

Solution. We use Newton's method to find the approximate value of the root.

Let $f(x) = x^4 - 3x + 1.$

Since the required root lies between 1 and 2 let $1 + h$ be the actual value of the root where $h = -\frac{f(1)}{f'(1)}$.

Now, $f(1) = -1$. Also $f'(x) = 4x^3 - 3$ and hence $f'(1) = 1.$

$$\therefore \quad h = -\left(\frac{-1}{1}\right) = 1.$$

$\therefore$ h is not small compared to the integral part of the root 1.

$\therefore$ We find the value of root still closer to 1.

We observe that $f(1.3) < 0$ and $f(1.4) > 0$ (verify).

Hence the root lies between 1.3 and 1.4.

$\therefore$ let $\alpha_1 = 1.3 + h_1$ be the actual value of the root.

$$\therefore \quad h_1 = -\frac{f(1.3)}{f'(1.3)} = -\left(\frac{-0.0439}{5.788}\right) = 0.008.$$

$\therefore$ $\alpha_1 + h_1 = 1.3 + 0.008.$

$= 1.31.$ (up to two places of decimals).

Now, $f'(x) = 3x^2 + 3$.

$\therefore \quad f'(0) = 3$.

$\therefore \quad h = \frac{1}{3} = 0.3$.

Hence the root is $0 + 0.3 = 0.3$ (second approximation)

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$0.3 + h_1$ where $h_1 = -\frac{f(0.3)}{f'(0.3)}$

$$\therefore \quad h_1 = \frac{(0.3)^3 + 3(0.3) - 1}{3(0.3)^2 + 3}$$

$= 0.022$.

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Problem 2. Find correct to 2 places of decimals the root of the equation $x^4 - 3x + 1$ that lies between 1 and 2.

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Let $f(x) = x^4 - 3x + 1$.

Since the required root lies between 1 and 2 let $1 + h$ be the actual value of the root where $h = -\frac{f(1)}{f'(1)}$.

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$\therefore \quad \alpha_1 + h_1 = 1.3 + 0.008$.

$= 1.31$. (up to two places of decimals).

3. Horner's method.

Horner's method is the most convenient way of finding approximate values of the irrational roots of the equation $f(x) = 0$ where $f(x)$ is a polynomial. The root is calculated in decimal form and the figures of the decimal are obtained in succession. We describe below the steps to be followed.

Step I Consider the equation $f(x) = 0$. Suppose this has a single root α in the interval $(a, a+1)$ where a is a positive integer. Then a can be located by using the condition that $f(a)$ and $f(a+1)$ are of opposite signs.

Step II. Suppose the exact value of the root is $a.a_1a_2$....... Diminish the roots of $f(x) = 0$ by a. Then we get the transformed equation $f_1(x) = 0$ having $0.a_1a_2$......... as a root.

Step III. Multiply the roots of $f_1(x) = 0$ by 10 and we obtain the transformed equation $f_2(x) = 0$ having $a_1.a_2a_3$..... as a root.

Step IV. By inspection we locate the root by finding two consective integr b and $b+1$ such that $f_2(b)$ and $f_2(b+1)$ are of opposite signs. The $b = a_1$ is the first decimal in the root making $a.a_1$ as the first approximatio of the root.

Repeat this process (steps I to IV) as many times as needed to g the roots of $f(x) = 0$ to any desired number of decimal places.

Problem 1. Find the positive root of $x^3 - x - 3 = 0$ correct to two plac of decimals.

Solution. Step I Let $f(x) = x^3 - x - 3$.

$f(1) = -3 < 0$ and $f(2) = 3 > 0$.

$\therefore$ A root lies between 1 and 2. Let the root be $1.a_1a_2$........

1	1	0	-1	-1
		1	1	0
	1	1	0	-3
		1	2	
	1	2	2	
		1		
	1	3		

$\therefore$ The transformed equation is $f_1(x) = x^3 + 3x^2 + 2x - 3 = 0$ and it has $0.a_1\, a_2 \ldots$, as a root.

Step III. Multiplying the roots of the equation $f_1(x) = 0$ by 10 we get the transformed equation as $f_2(x) = x^3 + 30x^2 + 200x - 3000 = 0$ and has a root $a_1 . a_1\, a_2 \ldots\ldots$

Step IV. (Location of roots $f_2(x) = 0$).

Now, $f_2(6) = -504 < 0$ and $f_2(7) = 213 > 0$ (verify)

$\therefore$ Hence $a_1 = 6$ and up to first order approximation the root is 1.6

Step V. Diminish the roots of $f_2(x)$ by 6.

$\therefore$ The transformed equation is $f_3(x) = x^3 + 48x^2 + 668x - 504 = 0$

[Refer table below]

6	1	30	200	−3000
		6	216	2496
	1	36	416	−504
		6	252	
	1	42	668	
		6		
	1	48		

VI. Multiplying the roots of $f_3(x) = 0$ by 10 we get

$f_4(x) = x^3 + 480x^2 + 66800x - 504000 = 0.$

VII. (Locations of roots of $f_4(x) = 0$).

As $f_4(7) = -12537 < 0$ and $f_4(8) = 61632 > 0$ (verify) Hence $a_2 = 7$.

VIII. Diminish the roots of $f_4(x)$ by 7.

7	1	480	66800	−504000
		7	3409	491463
	1	487	70209	12537
		7	3458	
	1	494	73667	
		7		
	1	501		

The transformed equation is $f_5(x) = x^3 + 501x^2 + 73667x - 12537 = 0$.

IX. Multiplying the roots of $f_5(x) = 0$ by 10 we get

$x^3 + 5010x^2 + 7366700x - 12537000 = 0$ and the root now is $a_2 . a_3 ..$

Step X. $a_3 = -\left(\frac{-12537000}{7366700}\right) = 1.70$ Hence $a_3 = 1$

Hence the required root is 1.671 and correct to 2 places of decimals, it is 1.67

			-1	-3	1.671
		0			
1	1		1	0	
		1			
				-3000 (root lies between 6 and 7)	6
			0		
	1	1			
			2	2496	
		1			
				-504000 (root lies between 7 and 8)	7
			200		
	1	2			
			216	491463	
		1			
				-12537000 (root lies between 1 and 2)	1
			416		
	1	30			
			252		
		6			
			66800		
6	1	36			
			3409		
		6			
			70209		
	1	42			
			3458		
		6			
			7366700		
	1	480			
		7			
7	1	487			
		7			
	1	494			
		7			
	1	5010			
	1				

Problem 1

Prove that

$$2^5 \cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

soln:

Let $x = \cos\theta + i\sin\theta$

Hence $x + \frac{1}{x} = 2\cos\theta$

$x^n + \frac{1}{x^n} = 2\cos n\theta$

now.

$$(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6$$

$x^5 \frac{1}{x}$ $x^4\left(\frac{1}{x^2}\right)$ $x^3 \frac{1}{x^3}$

$$2^6\cos^6\theta = x^6 + {}^6C_1 x^5\left(\frac{1}{x}\right) + 6C_2 x^4\left(\frac{1}{x}\right)^2 + {}^6C_3 x^3\left(\frac{1}{x^3}\right) + 6C_4 x^2\left(\frac{1}{x^4}\right) + 6\,{}^6C_5 x\left(\frac{1}{x^5}\right) + \frac{6C_6}{x}\left(\frac{1}{x^6}\right)$$

$$2^6\cos^6\theta = x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

1. Find the value of $2^8\cos^8\theta$

soln :

Let $x = \cos\theta + i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta$$

$$(x + 1/x)^n = 2\cos n\theta$$

now,

$$(2\cos\theta)^8 = (x + 1/x)^8$$

$$2^8\cos^8\theta = x^8 + 8C_1 x^7(1/x) + 8C_2 x^6 (1/x)^2 + 8C_3 x^5 (1/x)^3$$

$$+ 8C_4 x^4 (1/x)^4 + 8C_5 x^3 (1/x)^5 + 8C_6 x^2 (1/x)^6 +$$

$$8C_7 x (1/x)^7 + 8C_8 x^0 (1/x)^8$$

$$= x^8 + 8x^7 x^{-1} + \frac{8\times 7}{2\times 1} x^6 x^{-2} + \frac{8\times 7\times 6}{3\times 2\times 1} x^5 x^{-3} +$$

$$\frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1} x^4 x^{-4} + \frac{8\times 7\times 6\times 5\times 4}{5\times 4\times 3\times 2\times 1} x^3 x^{-5} +$$

$$\frac{8\times 7\times 6\times 5\times 4\times 3}{6\times 5\times 4\times 3\times 2\times 1} x^2 x^{-6} + \frac{8\times 7\times 6\times 5\times 4\times 3\times 2}{7\times 6\times 5\times 4\times 3\times 2\times 1} x x^{-7} + x^{-8}$$

$$= x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56x^{-2} + 28x^{-4} + 8x^{-6} + x^{-8}$$

$$= (x + 1/x)^8 + 8(x + 1/x)^6 + 28(x + 1/x)^4 + 56(x + 1/x)^2 +$$

$$= 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70$$

$$2^8\cos^8\theta = 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70$$

$$2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$$

Proble 3 :

Expand $\cos^5\theta \sin^3\theta$ in a series of sines of multiples of θ.

Soln.

Let $x = \cos\theta + i\sin\theta$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x + \frac{1}{x} = 2\cos\theta$$

$$x^n = \cos n\theta + i\sin n\theta$$

$$x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$i^3 = -i$

$$(2\cos\theta)^5(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^3$$

$$-2^8 \cos^5\theta\, i\sin^3\theta = \left(x+\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$$

$$= \left(x+\frac{1}{x}\right)^2\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^3$$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= x^6 - 3x^4\frac{1}{x^2} + 3x^2\frac{1}{x^4} - \frac{1}{x^6}$

$= x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}$

$$= \left(x+\frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x^2 + 2 + \frac{1}{x^2}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= x^8 - 3x^4 + \cancel{3} - \frac{1}{x^4} + 2x^6 - 6x^2 + \frac{6}{x^2} - \frac{2}{x^6} + x^4 - \cancel{3} + \frac{3}{x^4} - \frac{1}{x^8}$$

$$= \left(x^8 - \frac{1}{x^8}\right) + 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) - 6\left(x^2 - \frac{1}{x^2}\right)$$

$$= (2i\sin 8\theta) + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta)$$

$$= 2i\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

$$2i\left[-2^7\sin^3\theta\cos^5\theta\right] = 2i\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

$$\cos^5\theta\sin^3\theta = -\frac{1}{2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

Expansion of $\sin\theta$, $\cos\theta$, $\tan\theta$ in powers of θ

Theorem :

When θ is expressed in radians

(i) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

(ii) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$

(iii) $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots$

Proof (i)

Let $f(\theta) = \sin\theta$

The Taylor series expansion of f about the origin is given by $f(\theta) = f(0) + f'(0)\frac{\theta}{1!} + f''(0)\frac{\theta^2}{2!} + f'''(0)\frac{\theta^3}{3!} + \dots$

Now, $f(\theta) = \sin\theta$, Hence $f(0) = 0$

$f'(\theta) = \cos\theta$. Hence $f'(0) = 1$

$1! = 1$

$f''(\theta) = -\sin\theta$, Hence $f''(0) = 0$

$f'''(\theta) = -\cos\theta$, Hence $f'''(0) = -1$.

$$\therefore \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\cdots$$

Problem 3 : Show that if x is small.

$$\cos(\alpha+x) = \cos\alpha - x\sin\alpha - \frac{x^2}{2}\cos\alpha + \frac{x^3}{6}\sin\alpha.$$

Soln:

$$\cos(\alpha+x) = \cos\alpha\cos x - \sin\alpha\sin x$$

$$= \cos\alpha\left[1 - \frac{x^2}{2!} + \cdots\cdots\right] - \sin\alpha\left[x - \frac{x^3}{3!} + \cdots\cdots\right]$$

$$= \cos\alpha - x\sin\alpha - \frac{x^2}{2}\cos\alpha + \frac{x^3}{6}\sin\alpha$$

(Since x is small and neglecting higher powers of x)

Problem 4 : Solve approximately $\cos(\pi/3 + \theta) = 0.49$

Soln:

$$\cos(\pi/3 + \theta) = 0.49$$

$$\cos\pi/3\cos\theta - \sin\pi/3\sin\theta = 0.49$$

$$\frac{1}{2}\left[1 - \frac{\theta^2}{2!} + \cdots\cdots\right] - \frac{\sqrt{3}}{2}\left[\theta - \frac{\theta^3}{3!} + \cdots\cdots\right] = 0.49.$$

$$\frac{1}{2} - \frac{\sqrt{3}\theta}{2} = 0.49$$ (neglecting higher powers of θ)

$$\frac{\sqrt{3}\,\theta}{2} = \frac{1}{2} - 0.49 = \frac{1}{100}$$

$0.5 - 0.49 = 0.01 = \frac{1}{100}$

$\theta = \frac{1}{100} \times \frac{2}{\sqrt{3}}$

$$\therefore \theta = \frac{1}{50\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{150} = \frac{1.732}{150} = 0.0115 \text{ radian.}$$

$\frac{1732}{150 \times 1000} = \frac{346.4}{30 \times 10^3} = \frac{115.46}{10^4} = 0.011546$

$0.0115 \times 57.29 = 0.6588$

$0.6588 \times 60 = 39.53 = 40$ min

$$\therefore \theta = 40 \text{ minutes (approximately)}$$

Problem 5: Evaluate $\sin 3^\circ$ correct to three places of decimals.

soln:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \quad (\theta \text{ radian}).$$

$180^\circ = \pi$ radian. Hence $1^\circ = \pi/180$ radian.

$\pi/180 = 57.29$

$$\therefore 3^\circ = \frac{3\pi}{180} \text{ radian} = \frac{\pi}{60} \text{ radian.}$$

$$\sin 3^\circ = \sin(\pi/60) = \frac{\pi}{60} - \frac{1}{3!}\left(\frac{\pi}{60}\right)^3 + \ldots$$

$$= \frac{\pi}{60} \text{ (neglecting higher powers).}$$

3.414

$$= \left(\frac{22}{7}\right)\left(\frac{1}{60}\right) = 0.052.$$

Exam ✱✱ Problem 6: if $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \ldots$

and $y = 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \ldots$ pro that $x^2 = y$.

soln:

$$x = \frac{1+1}{1!} - \frac{3+1}{3!} + \frac{5+1}{5!} - \frac{7+1}{7!} + \ldots$$

$$= \left(\frac{1}{1!} - \frac{3}{3!} + \frac{5}{5!} - \frac{7}{7!} + \ldots\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots\right)$$

$$x = \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)$$

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$

$$x = \cos 1 + \sin 1$$

here $\theta = 1$

$\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

$$x^2 = (\cos 1 + \sin 1)^2$$

$$= \cos^2 1 + \sin^2 1 + 2\sin 1 \cos 1$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= 1 + \sin 2$$

$$= 1 + \left(\frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots\right)$$

$$x^2 = y$$

Hence the result.

Problem 3. prove that

$$\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

Soln:-

$$\tan 7\theta = \frac{7C_1\tan\theta - 7C_3\tan^3\theta + 7C_5\tan^5\theta - 7C_7\tan^7\theta}{1 - 7C_2\tan^2\theta + 7C_4\tan^4\theta - 7C_6\tan^6\theta}$$

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

Problem 4: prove that equation

$a^2\cos^2\theta + b^2\sin^2\theta + 2ag\cos\theta + 2bf\sin\theta + c = 0$ has four roots and then the sum of the four values of θ is an even multiple of π.

Soln:

$$a^2\cos^2\theta + b^2\sin^2\theta + 2ag\cos\theta + 2bf\sin\theta + c = 0.$$

$$a^2\left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)^2 + b^2\left(\frac{2\tan\theta/2}{1+\tan^2\theta/2}\right)^2 + 2ag\left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right).$$

$$+ 2bf\left(\frac{2\tan\theta/2}{1+\tan^2\theta/2}\right) + c = 0$$

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2 + b^2\left(\frac{2t}{1+t^2}\right)^2 + 2ag\left(\frac{1-t^2}{1+t^2}\right) + 2bf\left(\frac{2t}{1+t^2}\right) + c = 0$$

$$\left(\frac{1}{1+t^2}\right)^2\left[a^2(1-t^2)^2 + b^2 2^2 t^2 + 2ag(1-t^2)(1+t^2) + 2bf(2t) + c\right] = 0$$

$$2ag[(1^2-t^4)-t]$$

$$\left(\frac{1}{1+t^2}\right)^2\left[a^2(1+t^4-2t^2) + b^2 2^2 t^2 + 2ag(1^2-t^4) + 2bf(2t)(1+t^2) + \frac{c}{(1+t^2)^2}\right] = 0$$

$$\left(\frac{1}{1+t^2}\right)^2 a^2 + a^2t^4 - 2a^2t^2 + 2^2b^2t^2 + 2ag((1-t^4)-t)$$

$$+ 2bf(2t+2t^3) + c(1+t^4+2t^2)] = 0.$$

$$\left(\frac{1}{1+t^2}\right)^2 a^2 + a^2t^4 - 2a^2t^2 + 4b^2t^2 + 2ag - 2agt^4 + 4bft + 4bft^3$$

$$+ c + ct^4 + 2ct^2] = 0$$

$$\left(\frac{1}{1+t^2}\right)^2\left[t^4(a^2-2ag+c) + t^3(4bf) + t^2(4b^2-2a^2+2c)\right.$$

$$\left. + 4bft + (a^2+2ag+c)\right] = 0$$

$$(a^2-2ag+c)t^4 + 4bf\,t^3 + (4b^2 - 2a^2 + 2c)\,t^2 + 4bf\,t + (a^2+2ag+c) = 0$$

This is a fourth degree equation in t and has four roots say,

$$t_i = \tan\left(\frac{\theta_i}{2}\right) \; ; \quad i = 1,2,3,4$$

Now,

$$\tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4}$$

$$S_1 = \Sigma \tan\left(\frac{\theta_1}{2}\right) = \Sigma t_1 = \frac{-4bf}{a^2-2ag+c}$$

$$S_3 = \Sigma \tan\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\theta_2}{2}\right) \tan\left(\frac{\theta_3}{2}\right) = \Sigma t_1 t_2 t_3 = \frac{-4bf}{a^2-2ag+c}$$

$$\tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}\right) = 0$$

$$\therefore \frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} = \cancel{\tan}\, n\pi \quad \text{where } n \in \mathbb{Z}$$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2n\pi \quad \text{Where } n \in \mathbb{Z}$$

Hence the result proved

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proof (ii) :

Let $f(\theta) = \tan\theta$. Hence $f(0) = 0$.

$\sec^2\theta = (1+\tan^2\theta)$

$f'(\theta) = \sec^2\theta$. Hence $f'(0) = 1$

$f''(\theta) = 2\sec^2\theta\tan\theta = 2(\tan\theta + \tan^3\theta)$, Hence $f''(0) = 0$

$f'''(\theta) = 2(\sec^2\theta + 3\tan^2\theta\sec^2\theta)$ $\quad 3\tan^2\theta(1+\tan^2\theta)$

$= 2(3\tan^4\theta + 4\tan^2\theta + 1)$, Hence $f'''(0) = 2$

$$f\ (\theta) = 2(12\tan^3\theta\sec^2\theta + 8\tan\theta\sec^2\theta)$$

$$= 8(3\tan^5\theta + 5\tan^3\theta + 2\tan\theta), \text{ Hence } f'''(0) = 0$$

$$f^{(iv)}(\theta) = 8(15\tan^4\theta\sec^2\theta + 15\tan^2\theta\sec^2\theta + 2\sec^2\theta),$$

$$\text{Hence } f^{(v)}(0) = 16$$

$$\therefore \tan\theta = 0 + \frac{\theta}{1!} + 0 + \frac{2\theta^3}{3!} + 0 + \frac{16\theta^5}{5!} + \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

Solved Problems.

Problem 1:

Find approximately the value of θ in radians if $\frac{\sin\theta}{\theta} = \frac{863}{864}$

Soln:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{863}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$1 - \frac{1}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\therefore \frac{1}{864} = \frac{\theta^2}{3!} - \frac{\theta^4}{5!}$$

$$\therefore \frac{\theta^2}{3!} = \frac{1}{864}$$ (neglecting higher powers of θ)

$$\theta^2 = \frac{6}{864} = \frac{1}{144}$$

$$\therefore \theta = \frac{1}{12} \text{ radians}$$

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problem 2 :

If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$ show that θ is approximately equal to $1^\circ 58'$.

<u>Soln</u> :

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

$$\frac{\tan\theta}{\theta} = 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \cdots$$

$$\frac{2524}{2523} = 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \cdots$$

$$1 + \frac{1}{2523} = 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \cdots$$

$$\frac{1}{2523} = \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \cdots$$

$$\frac{\theta^2}{3} = \frac{1}{2523} \text{ (neglecting higher powers of } \theta)$$

$$\theta^2 = \frac{3}{2523} = \frac{1}{841}$$

$$\theta = \frac{1}{29} \text{ radians.}$$

$$= \frac{1}{29} \times 57.29 \text{ degrees approximately.}$$

1.977   97 × 60 = 5820

Hence $\theta = 1^\circ 58'$ (approximately).

Exercises :

1. Find the value of θ when

(i) $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$.

Soln: $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\frac{2165}{2166} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\cancel{1} - \frac{1}{2166} = \cancel{1} - \frac{\theta^2}{3!} + \frac{\theta^4}{5!}$$

$$\frac{1}{2166} = \frac{\theta^2}{3!} - \frac{\theta^4}{5!}$$

$$\frac{\theta^2}{3!} = \frac{1}{2166} \text{ (neglecting higher powers of } \theta)$$

$$\theta^2 = \frac{\cancel{6}^{1}}{\cancel{2166}_{361}} = \frac{1}{361}$$

$$\begin{array}{r} 19 \\ 19 \\ \hline 171 \\ 19 \\ \hline 361 \end{array}$$

$$\therefore \theta = \frac{1}{19}$$

(ii) $\frac{\sin\theta}{\theta} = \frac{1013}{1014}$

Soln: $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\frac{1013}{1014} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\cancel{1} - \frac{1}{1014} = \cancel{1} - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\frac{1}{1014} = \frac{\theta^2}{3!} - \frac{\theta^4}{5!}.$$

$$\frac{\theta^2}{3!} = \frac{1}{1014} \quad \text{(neglecting higher powers of } \theta)$$

$$\theta^2 = \frac{\cancel{6}^{\,1}}{\cancel{1014}_{\,169}} = \frac{1}{169}$$

$$\begin{array}{r} 17 \\ 17 \\ \hline 119 \\ 17 \\ \hline 289 \end{array} \qquad \begin{array}{r} 13 \\ 13 \\ \hline 39 \\ 13 \\ \hline 169 \end{array}$$

$$\therefore \theta = \frac{1}{13}$$

(iii) $\dfrac{\sin\theta}{\theta} = \dfrac{19493}{19494}$

Soln:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\frac{19493}{19494} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$1 - \frac{1}{19494} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \dots$$

$$\frac{1}{19494} = \frac{\theta^2}{3!} - \frac{\theta^4}{5!} \quad \text{(neglecting higher powers of } \theta)$$

$$\frac{\theta^2}{3!} = \frac{1}{19494}$$

$$\begin{array}{r} 57 \\ 57 \\ \hline 399 \\ 285 \\ \hline 3249 \end{array} \qquad \begin{array}{r} 37 \\ 37 \\ \hline 259 \\ 111 \\ \hline 1369 \end{array}$$

$$\theta^2 = \frac{\cancel{6}^{\,1}}{\cancel{19494}_{\,3249}} = \frac{1}{3249}$$

$$\therefore \theta = \frac{1}{57}$$

(iv) $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$

Soln:
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{5045}{5046} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$1 - \frac{1}{5046} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{1}{5046} = \frac{\theta^2}{3!} - \frac{\theta^4}{5!}.$$

$$\frac{\theta^2}{3!} = \frac{1}{5046}$$ [neglecting higher powers of θ]

$$\theta^2 = \frac{\cancel{6}\,1}{\cancel{5046}\,841} = \frac{1}{841}$$

$$\begin{array}{r} 19 \\ \times\ 19 \\ \hline 171 \\ 19 \\ \hline 361 \end{array} \qquad \begin{array}{r} 29 \\ \times\ 29 \\ \hline 261 \\ 58 \\ \hline 841 \end{array}$$

$$\therefore \theta = \frac{1}{29}$$

problem 7: If θ is small prove that $\theta \cot\theta = 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}$ approximately.

Soln: $\theta\cot\theta = \frac{\theta}{\tan\theta}$ $\qquad \cot\theta = \frac{1}{\tan\theta}$

$$= \frac{\theta}{\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}}$$ (neglecting higher powers of θ)

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

$$\theta\cot\theta = \frac{\cancel{\theta}}{\cancel{\theta}}\left(\frac{1}{1+\frac{\theta^2}{3}+\frac{2\theta^4}{15}}\right)$$ take out (θ)

$$= \left(1 + \underbrace{\frac{\theta^2}{3}+\frac{2\theta^4}{15}}_{}\right)^{-1} \qquad (1+x)^{-1}$$

$$\boxed{(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots}$$

$$= 1 - \left(\frac{\theta^2}{3} + \frac{2\theta^4}{15}\right) + \left(\frac{\theta^2}{3} + \frac{2\theta^4}{15}\right)^2 - \cdots\cdots$$

(neglecting higher powers of θ)

$$= 1 - \frac{\theta^2}{3} - \left(\frac{2}{15} - \frac{1}{9}\right)\theta^4 \text{ (approximately)}$$

$$= 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45} \text{ (approximately)}$$

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problem 8: Prove that when θ is small

$$\frac{1}{6}\sin^3\theta = \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + (1+3^2+3^4)\frac{\theta^7}{7} \cdots\cdots$$

Soln: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore 4\sin^3\theta = 3\sin\theta - \sin 3\theta$$

$$= 3\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right] - \left[3\theta - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!} - \frac{(3\theta)^7}{7!} + \cdots\right]$$

$$= \left[\theta - \frac{\theta^3}{3!}(3-3^3) + \frac{\theta^5}{5!}(3-3^5) - \frac{\theta^7}{7!}(3-3^7) + \dots\right]$$

$$= 3\left[(3^2-1)\frac{\theta^3}{3!} - (3^4-1)\frac{\theta^5}{5!} + (3^6-1)\frac{\theta^7}{7!} - \dots\right]$$

$(3^2-1)(3^2+1)$ $\quad a^3-b^3 = (a-b)(a^2+ab+b^2)$

$$= 3(3^2-1)\left[\frac{\theta^3}{3!} - (3^2+1)\frac{\theta^5}{5!} + (3^4+3^2+1)\frac{\theta^7}{7!} - \dots\right]$$

$$\therefore \sin^3\theta = 6\left[\frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + (1+3^2+3^4)\frac{\theta^7}{7!} - \dots\right]$$

Hence the result.

Problem 9. Show that $\lim\limits_{x\to 0} \dfrac{3\sin x - \sin 3x}{x - \sin x} = 24$.

Soln:

$$\lim_{x\to 0} \frac{3\sin x - \sin 3x}{x - \sin x}$$

$$= \lim_{x\to 0}\left[\frac{3\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - \left(\frac{3x}{1!} - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots\right)}{x - \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\left(\frac{-x^3}{2} + \frac{27x^3}{6}\right) - \left(\frac{3x^5}{5!} - \frac{(3x)^5}{5!}\right) + \dots}{\frac{x^3}{3!} - \frac{x^5}{5!} + \dots}\right]$$

take out x^3

$$= \lim_{x\to 0}\left[\frac{\left(-\frac{1}{2} + \frac{9}{2}\right) - \left(\frac{3x^2}{5!} - \frac{3^5x^2}{5!}\right) + \dots}{\frac{1}{3!} - \frac{x^2}{5!} + \dots}\right] = 24.$$

$$= \frac{-\frac{1}{2} + \frac{9}{2}}{\frac{1}{3!}} =$$

Problem 10: Show that $\lim\limits_{x\to 0}\left(\dfrac{\cos^2 ax-\cos^2 bx}{1-\cos cx}\right) = \dfrac{2(b^2-a^2)}{c^2}$

Soln:

$$\lim_{x\to 0}\left(\frac{\cos^2 ax-\cos^2 bx}{1-\cos cx}\right) = \lim_{x\to 0}\left[\frac{\left(1-\frac{a^2x^2}{2!}+\dots\right)^2-\left(1-\frac{b^2x^2}{2!}+\dots\right)^2}{\cancel{1}-\left(\cancel{1}-\frac{c^2x^2}{2!}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\left(\cancel{1}-\frac{2a^2x^2}{2!}+\dots\right)-\left(\cancel{1}-\frac{2b^2x^2}{2!}+\dots\right)}{c^2x^2/2}\right]$$

$$= \lim_{x\to 0}\left(\frac{(b^2-a^2)x^2+\dots}{\frac{c^2x^2}{2}+\dots}\right) = 2\,\frac{(b^2-a^2)}{c^2}$$

$$\lim_{x\to 0}\left(\frac{\cos^2 ax-\cos^2 bx}{1-\cos cx}\right) = \frac{2(b^2-a^2)}{c^2}$$

Problem 11: Evaluate $\lim\limits_{x\to \pi/2}\left(\dfrac{\sin x+\cos 2x}{\cos^2 x}\right)$.

Soln:

put $x=\theta+\pi/2$. As $x\to\pi/2$ we notice $\theta\to 0$.

$$\lim_{x\to\pi/2}\left(\frac{\sin x+\cos 2x}{\cos^2 x}\right) = \lim_{\theta\to 0}\left(\frac{\cos\theta-\cos 2\theta}{\sin^2\theta}\right)$$

$$= \lim_{\theta\to 0}\left[\frac{\left(\cancel{1}-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots\right)-\left(\cancel{1}-\frac{4\theta^2}{2!}+\frac{16\theta^4}{4!}-\dots\right)}{\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\dots\right)^2}\right]$$

$$= \lim_{\theta\to 0}\left(\frac{\frac{3}{2}\theta^2-\frac{5}{8}\theta^4+\dots}{\theta^2\left(1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\dots\right)}\right) = \lim_{\theta\to 0}\left(\frac{\theta^2\left(\frac{3}{2}-\frac{5}{\dots}\right.}{\theta^2\left(1-\frac{\theta}{3!}+\dots\right.}\right.$$

$$= 3/2$$

4) Prove the following

(i) $2^3\cos^4\theta = \cos^4\theta + 4\cos 2\theta + 3$

Soln:

$$x = \cos\theta + i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$

now,

$$(2\cos\theta)^4 = \left(x + \frac{1}{x}\right)^4$$

$$2^4\cos^4\theta = x^4 + 4C_1 x^3\left(\frac{1}{x}\right) + 4C_2 x^2\left(\frac{1}{x}\right)^2 + 4C_3 x\left(\frac{1}{x}\right)^3 + 4C_4\left(\frac{1}{x}\right)^4$$

$$= x^4 + \frac{4x^3}{x} + \frac{6x^2}{x^2} + \frac{4x}{x^3} + \frac{1}{x^4}$$

$$= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$

$$= \left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 6$$

$$= (2\cos 4\theta) + 4(2\cos 2\theta) + 6$$

$$2^4\cos^4\theta = 2(\cos 4\theta + 2\cos 2\theta + 3)$$

$$\frac{2^4\cos^4\theta}{2} = \cos 4\theta + 4\cos 2\theta + 3$$

$$2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3.$$

$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$

now $(2\cos\theta)^5 = (x + \frac{1}{x})^5$

$2^5\cos^5\theta = x^5 + 5C_1 x^4\left(\frac{1}{x}\right) + 5C_2 x^3\left(\frac{1}{x}\right)^2 +$

$5C_3 x^2\left(\frac{1}{x}\right)^3 + 5C_4 x\left(\frac{1}{x}\right)^4 + 5C_5\left(\frac{1}{x}\right)^5$

$= x^5 + \frac{5x^4}{x} + \frac{10x^3}{x^2} + \frac{10x^2}{x^3} + \frac{5x}{x^4} + \frac{1}{x^5}$

$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

$= \left(x^5 + \frac{1}{x^5}\right) + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$

$= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$2^5\cos^5\theta = 2\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$

$\frac{2^5\cos^5\theta}{2} = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$

$2^4\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta.$

(iii) $2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$

Soln:

$x = \cos\theta + i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$

now,

$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7$

$$2^7\cos^7\theta = x^7 + 7C_1 x^6\left(\frac{1}{x}\right) + 7C_2 x^5\left(\frac{1}{x}\right)^2 + 7C_3 x^4\left(\frac{1}{x}\right)^3$$

$$+ 7C_4 x^3\left(\frac{1}{x}\right)^4 + 7C_5 x^2\left(\frac{1}{x}\right)^5 + 7C_6 x\left(\frac{1}{x}\right)^6$$

$$+ 7C_7\left(\frac{1}{x}\right)^7$$

$$= x^7 + \frac{7x^6}{x} + \frac{21x^5}{x^2} + \frac{35x^4}{x^3} + \frac{35x^3}{x^4} + \frac{21x^2}{x^5} + \frac{7x}{x^6} + \frac{1}{x^7}$$

$$= x^7 + 7x^5 + 21x^3 + 35x + 35/x + 21/x^3 + 7/x^5 + 1/x^7$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + 1/x\right)$$

$$= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$$

$$2^7\cos^7\theta = 2(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$$

$$\frac{2^7\cos^7\theta}{2} = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$$

$$2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$$

(vii) $\sin^6\theta = -(1/2^5)\left(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right)$

Soln:

$$x = \cos\theta + i\sin\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

now,

$$(2i\sin\theta)^6 = \left(x - 1/x\right)^6$$

$$2^6 i^6 \sin^6\theta = x^6 - 6C_1 x^5\left(1/x\right) + 6C_2 x^4\left(1/x\right)^2 - 6C_3 x^3\left(1/x\right)^3$$

$$+ 6C_4 x^2\left(\frac{1}{x}\right)^4 - 6C_5 x\left(\frac{1}{x}\right)^5 + 6C_6\left(\frac{1}{x}\right)^6$$

$$-2^6\sin^6\theta = x^6 - \frac{6x^5}{x} + \frac{15x^4}{x^2} - \frac{20x^3}{x^3} + \frac{15x^2}{x^4} - \frac{6x}{x^5} + \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$= 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$-2^6\sin^6\theta = 2\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]$$

$$\frac{-2^6\sin^6\theta}{2} = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10$$

$$-2^5\sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10$$

$$\sin^6\theta = -\left(\frac{1}{2^5}\right)\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]$$

---

viii) $\sin^9\theta = \frac{1}{256}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$

Soln:

$$x = \cos\theta + i\sin\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

now,

$$(2i\sin\theta)^9 = \left(x - \frac{1}{x}\right)^9$$

$$2^9 i^9 \sin^9\theta = x^9 - 9C_1 x^8\left(\frac{1}{x}\right) + 9C_2 x^7\left(\frac{1}{x}\right)^2 - 9C_3 x^6\left(\frac{1}{x}\right)^3 +$$

$$9C_4 x^5\left(\frac{1}{x}\right)^4 - 9C_5 x^4\left(\frac{1}{x}\right)^5 + 9C_6 x^3\left(\frac{1}{x}\right)^6 - 9C_7 x^2\left(\frac{1}{x}\right)^7$$

$$+ 9C_8\, x\left(\frac{1}{x}\right)^8 - 9C_9\left(\frac{1}{x}\right)^9$$

$$= x^9 - \frac{9x^8}{x} + \frac{36x^7}{x^2} - \frac{84x^6}{x^3} + \frac{126x^5}{x^4} - \frac{126x^4}{x^5} + \frac{84x^3}{x^6}$$
$$- \frac{36x^2}{x^7} + \frac{9x}{x^8} - \frac{1}{x^9}$$

$$= x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x} + \frac{84}{x^3} - \frac{36}{x^5}$$
$$+ \frac{9}{x^7} - \frac{1}{x^9}$$

$$= \left(x^9 - \frac{1}{x^9}\right) - 9\left(x^7 - \frac{1}{x^7}\right) + 36\left(x^5 - \frac{1}{x^5}\right) - 84\left(x^3 - \frac{1}{x^3}\right)$$
$$+ 126\left(x - \frac{1}{x}\right)$$

$$= (2i\sin 9\theta) - 9(2i\sin 7\theta) + 36(2i\sin 5\theta) - 84(2i\sin 3\theta)$$
$$+ 126(2i\sin\theta)$$

$$2^9 i^9 \sin^9\theta = 2i\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$$

$$\frac{2^9 i^9 \sin^9\theta}{2i} = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta$$

$i^6 = -1$

$i^8 = (-1) \times i^2$

$= -1 \times -1$

$i^8 = 1$

$$2^8 i^8 \sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta$$

$$2^8 \sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta$$

$$\sin^9\theta = \frac{1}{2^8}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$$

$$\sin^9\theta = \frac{1}{256}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$$

2) Prove the following

(i) $\sin^3\theta \cos\theta = -\left(\frac{1}{8}\right)(\sin 4\theta - 2\sin 2\theta)$

Soln:

$$x = \cos\theta + i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

now,

$$(2i\sin\theta)^3 2\cos\theta = \left(x - \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)$$

$$2^3 i^3 \sin^3\theta . 2\cos\theta = \left[a^3 - 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x^2}\right) - \frac{1}{x^3}\right]\left[x + \frac{1}{x}\right]$$

$$= \left[x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}\right]\left[x + \frac{1}{x}\right]$$

$$= x^4 - 3x^2 + \cancel{3} - \frac{1}{x^2} + x^2 - \cancel{3} + \frac{3}{x^2} - \frac{1}{x^4}$$

$$= \left(x^4 - \frac{1}{x^4}\right) - 2\left(x^2 - \frac{1}{x^2}\right)$$

$$= 2i\sin 4\theta - 2(2i\sin 2\theta)$$

$$2^3 i^3 \sin^3\theta . 2\cos\theta = 2i\left[\sin 4\theta - 2\sin 2\theta\right]$$

$$\frac{2^3 i^3 \sin^3\theta . 2\cos\theta}{2i} = \sin 4\theta - 2\sin 2\theta$$

$$2^3 i^2 \sin^3\theta . \cos\theta = \sin 4\theta - 2\sin 2\theta$$

$$-2^3 \sin^3\theta . \cos\theta = \sin 4\theta - 2\sin 2\theta$$

$$\sin^3\theta\cos\theta = -\frac{1}{2^3}\left[\sin 4\theta - 2\sin 2\theta\right]$$

$$\sin^3\theta\cos\theta = -\left(\frac{1}{8}\right)\left[\sin 4\theta - 2\sin 2\theta\right]$$

(ii) $$\cos^5\theta\sin^4\theta = \frac{1}{2^8}\left[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta\right]$$

Soln:

$$x = \cos\theta + i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$(2\cos\theta)^5(2i\sin\theta)^4 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^4$$

$$2^5\cos^5\theta \cdot 2^4 i^4\sin^4\theta = \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^4\right]$$

$$2^9\cos^5\theta \cdot i^4\sin^4\theta = \left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^4$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

$$= \left(x+\frac{1}{x}\right)\left[x^8 - 4x^6\left(\frac{1}{x^2}\right) + 6x^4\left(\frac{1}{x^4}\right) - 4x^2\left(\frac{1}{x^6}\right) + \frac{1}{x^8}\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^8 - 4x^4 + 6 - \frac{4}{x^4} + \frac{1}{x^8}\right]$$

$$= x^9 - 4x^5 + 6x - \frac{4x}{x^4} + \frac{x}{x^8} + \frac{x^3}{x} - \frac{4x^4}{x} + \frac{6}{x} - \frac{4}{x^5} + \frac{1}{x^9}$$

$$2^9 \cos^5\theta \sin^4\theta = \left(x^9 + \frac{1}{x^9}\right) + \left(x^7 + \frac{1}{x^7}\right) - 4\left(x^5 + \frac{1}{x^5}\right) - 4\left(x^3 + \frac{1}{x^3}\right) + 6\left(x + \frac{1}{x}\right)$$

$$= 2\cos 9\theta + 2\cos 7\theta - 4(2\cos 5\theta) - 4(2\cos 3\theta) + 6(2\cos\theta)$$

$$= 2\left[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta\right]$$

$$\frac{2^9 \cos^5\theta \sin^4\theta}{2} = \cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta$$

$$2^8 \cos^5\theta \sin^4\theta = \cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta$$

$$\cos^5\theta \sin^4\theta = \frac{1}{2^8}\left(\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta\right)$$

(iii) $2^5 \sin^4\theta \cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

Soln:

$$x = \cos\theta + i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$(2i\sin\theta)^4 (2\cos\theta)^2 = \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^2$$

$$2^4 i^4 \sin^4\theta . 2^2 \cos^2\theta = \left[x^4 - 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x^2}\right) - 4x\left(\frac{1}{x^3}\right) + \left(\frac{1}{x^4}\right)\right] \left[x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}\right]$$

$$= \left[x^4 + \frac{1}{x^4} - 4x^2 + 6 - \frac{4}{x^2}\right]\left[x^2 + \frac{1}{x^2} + 2\right]$$

$$= x^6 + \frac{1}{x^2} - 4x^4 + 6x^2 - 4 + x^2 + \frac{1}{x^6} - 4 + \frac{6}{x^2} - \frac{4}{x^4}$$

$$+ 2x^4 + \frac{2}{x^4} - 8x^2 + 12 - \frac{8}{x^2}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + \frac{1}{x^4}\right) - \left(x^2 + \frac{1}{x^2}\right) + 4$$

$$2^6 i^4 \sin^4\theta \cos^2\theta = 2\cos 6\theta - 2(2\cos 4\theta) - 2\cos 2\theta + 4$$

$i^4 = 1$

$$2^6 \sin^4\theta \cos^2\theta = \cancel{2}\left[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\right]$$

$$2^5 \sin^4\theta \cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$

(iv) $2^6 \sin^4\theta \cos^3\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$

Soln:

$$x = \cos\theta + i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

now,

$$(2i\sin\theta)^4 (2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$$

$$2^4 i^4 \sin^4\theta \, 2^3 \cos^3\theta = \left(x - \frac{1}{x}\right)\left[\left(x - \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)^3\right]$$

$$2^7 i^4 \sin^4\theta \cos^3\theta = \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x - \frac{1}{x}\right)\left[x^6 - 3x^4\left(\frac{1}{x^2}\right) + 3x^2\left(\frac{1}{x^4}\right) - \frac{1}{x^6}\right]$$

$$=\left(x-\frac{1}{x}\right)\left[x^6-3x^2+\frac{3}{x^2}-\frac{1}{x^6}\right]$$

$$=x^7-3x^3+\frac{3}{x}-\frac{1}{x^5}-x^5+3x-\frac{3}{x^3}+\frac{1}{x^7}$$

$$=\left(x^7+\frac{1}{x^7}\right)-\left(x^5+\frac{1}{x^5}\right)-3\left(x^3+\frac{1}{x^3}\right)+3\left(x+\frac{1}{x}\right)$$

$$2^7\sin^4\theta\cos^3\theta=2\cos7\theta-2\cos5\theta-3(2\cos3\theta)+3(2\cos\theta)$$

$$2^7\sin^4\theta\cos^3\theta=2\left[\cos7\theta-\cos5\theta-3\cos3\theta+3\cos\theta\right]$$

$$2^6\sin^4\theta\cos^3\theta=\cos7\theta-\cos5\theta-3\cos3\theta+3\cos\theta$$

$$2^8\cos^5\theta\sin^4\theta=\cos9\theta+\cos7\theta-4\cos5\theta-4\cos3\theta+6\cos\theta$$

soln:

$$x=\cos\theta+i\sin\theta$$

$$x+\frac{1}{x}=2\cos\theta \qquad x^n+\frac{1}{x^n}=2\cos n\theta$$

$$x-\frac{1}{x}=2i\sin\theta \qquad x^n-\frac{1}{x^n}=2i\sin n\theta$$

$$(2i\sin\theta)^4(2\cos\theta)^5=\left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^4$$

$$2^5\cos^5\theta\,2^4i^4\sin^4\theta=\left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^4\right]$$

$$=\left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^4$$

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

$$=\left(x+\frac{1}{x}\right)\left[x^8-4x^6\left(\frac{1}{x^2}\right)+6x^4\left(\frac{1}{x^4}\right)-4x^2\left(\frac{1}{x^6}\right)+\frac{1}{x^8}\right]$$

$$= \left(x + \frac{1}{x}\right)\left[x^8 - 4x^4 + 6 - \frac{4}{x^4} + \frac{1}{x^8}\right]$$

$$= x^9 - 4x^5 + 6x - \frac{4}{x^3} + \frac{1}{x^7} + x^7 - 4x^3 + \frac{6}{x} - \frac{4}{x^5} + \frac{1}{x^9}$$

$$= \left(x^9 + \frac{1}{x^9}\right) + \left(x^7 + \frac{1}{x^7}\right) - 4\left(x^5 + \frac{1}{x^5}\right) - 4\left(x^3 + \frac{1}{x^3}\right) + 6\left(x + \frac{1}{x}\right)$$

$$2^9 \cos^5\theta \sin^4\theta = 2\cos 9\theta + 2\cos 7\theta - 4(2\cos 5\theta) - 4(2\cos 3\theta) + 6(2\cos\theta)$$

$$2^9 \cos^5\theta \sin^4\theta = \cancel{2}\left[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta\right]$$

$$2^8 \cos^5\theta \sin^4\theta = \cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta$$

CHAPTER - 2

Unit - V

CHAPTER -2 Unit - V

HYPERBOLIC FUNCTION

Hyperbolic functions:

Definitions, The hyperbolic functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \; ; \quad \cosh x = \frac{e^x + e^{-x}}{2} \; ;$$

$$\tanh x = \frac{\sinh x}{\cosh x} \; ; \quad \coth x = \frac{\cosh x}{\sinh x} \; ;$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} \; ; \quad \operatorname{sech} x = \frac{1}{\cosh x} .$$

Note: The following are immediate consequences of the definition

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

we observe that cosh $x \geq 1$ for all x.

3. $\cosh 0 = 1$ and $\sinh 0 = 0$

Result 1. $\cosh^2 x - \sinh^2 x = 1$

proof:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{(e^x+e^{-x})^2}{4} - \frac{(e^x-e^{-x})^2}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = 1$$

Result 2. $\sinh 2x = 2 \sinh x \cosh x$.

proof:

$$2 \sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x.$$

Result 3. $\cosh^2 x + \sinh^2 x = \cosh 2x$

proof:

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^x+e^{-x})^2}{4} + \frac{(e^x-e^{-x})^2}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4} = \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{2(e^{2x}+e^{-2x})}{4} = \frac{e^{2x}+e^{-2x}}{2}$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$$

Result 4. From result (1) and (3) we get the following result.

result (1) + (3) = $2\cosh^2 x - 1 = \cosh 2x$

$\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = 1 + 2\sinh^2 x$

① + ② $\Rightarrow$ $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

① − ② $\Rightarrow$ $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Relation between hyperbolic functions and circular trigonometric functions.

Theorem, (i) $\sin(ix) = i\sinh x$

Exam (ii) $\cos(ix) = \cosh x$ 2M

(iii) $\tan(ix) = i\tanh x$

proof: We know $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \cdots$

put $\theta = ix$. Then we have

(i) $\sin(ix) = (ix) - \dfrac{i^3x^3}{3!} + \dfrac{i^5x^5}{5!} - \cdots$

$= i\left(x - \dfrac{i^2x^3}{3!} + \dfrac{i^4x^5}{5!}\right)$

$= i\left(x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots\right) = i\sinh x$

proof for (ii) is similar to (i)

(iii) $\tan(ix) = \dfrac{\sin(ix)}{\cos(ix)} = \dfrac{i\sinh x}{\cosh x} = i\tanh x$

★ Corollary. (i) $\sinh x = (1/i)\sin(ix) = -i\sin(ix)$

(ii) $\cosh x = \cos ix$

(iii) $\tanh x = -i\tan(ix)$.

Note 1. Using the above relations we can derive relations between hyperbolic functions corresponding to relations between circular trigonometric functions.

Example 1.

Corresponding to the formula $\cos^2\theta + \sin^2\theta = 1$ we have

$\cosh^2 x - \sinh^2 x = 1$

put $\theta = ix$ in $\cos^2\theta + \sin^2\theta = 1$

Then $\cos^2(ix) + \sin^2(ix) = 1$

$$\Rightarrow \cosh^2 x + (i\sinh x)^2 = 1$$

$$\Rightarrow \cosh^2 x - \sinh^2 x = 1.$$

Example 2. X

Consider $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

put $A = ix$ and $B = iy$

Then $\cos i(x+y) = \cos(ix)\cos(iy) - \sin(ix)\sin(iy)$ $\cos(ix+iy)$

$$\cosh(x+y) = \cosh x \cosh y - (i\sinh x)(i\sinh y)$$

---

proof (ii)

We know $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$ put $\theta = ix$

$$\cos ix = 1 - \frac{i^2x^2}{2!} + \frac{i^4x^4}{4!} - \dots$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$\cos(ix) = \cosh x$.

Inverse Hyperbolic Functions.

Consider the function $y = \sinh x$ this is a 1-1 onto map from $R \to R$.

Given any $y \in R$ there exists unique x such that $\sinh x = y$. we define $x = \sinh^{-1} y$.

Similarly, $y = \cosh x$ is a map from $R \to [1, \infty)$.

Both x and $(-x)$ have the same image under $\cosh x$.

Hence given any $y \in [1, \infty)$, We can find unique positive x such that $\cosh x = y$.

We define $x = \cosh^{-1} y$ and x is called the Principal value of $\cosh^{-1} y$.

The function $y = \tanh x$ is a map from $R \to (-1, 1)$.

Given any $y \in R$ there exists unique x such that $\tanh x = y$.

We define $x = \tanh^{-1} y$.

Theorem, (i) $\sinh^{-1} x = \log_e (x + \sqrt{x^2+1})$

(ii) $\cosh^{-1} x = \log_e (x + \sqrt{x^2-1})$

(iii) $\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

Proof: (i) Let $y = \sinh^{-1} x$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$\times e^y \Rightarrow$ $$2xe^y = e^{2y} - e^0$$

$$2xe^y = e^{2y} - 1$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= \frac{2\left[x \pm \sqrt{x^2+1}\right]}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$y = \log_e (x + \sqrt{x^2+1})$$

$$\sinh^{-1} x = \log_e (x + \sqrt{x^2+1})$$

(ii) Let $y = \cosh^{-1} x$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + e^{-y}$$

$$2xe^y = e^{2y} + 1$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{2\left[x \pm \sqrt{x^2 - 1}\right]}{2}$$

$$e^y = x \pm \sqrt{x^2 - 1}$$

$$= x + \sqrt{x^2 - 1} \quad \text{(or)} \quad x - \sqrt{x^2 - 1}$$

$$= x + \sqrt{x^2 - 1} \quad \text{(or)} \quad \frac{1}{x + \sqrt{x^2 - 1}}$$

$$y = \log_e\left(x + \sqrt{x^2 - 1}\right) \quad \text{or} \quad -\log_e\left(x + \sqrt{x^2 - 1}\right)$$

$$y = \pm \log_e\left(x + \sqrt{x^2 - 1}\right)$$

Hence the principal value of $\cosh^{-1} x$ is given by

$$\cosh^{-1} x = \log_e\left(x + \sqrt{x^2 - 1}\right)$$

$\tanh y = \frac{\sinh}{\cosh}$

$= \frac{e^y - e^{-y}/2}{e^y + e^{-y}/2}$

~~Let~~ (iii) Let $y = \tanh^{-1} x$.

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$= \frac{e^y - e^{-y}}{e^y + e^{-y}}$

$$e^{y} - e^{-y} = x(e^{y} + e^{-y})$$

$$e^{y}(1-x) = e^{-y}(1+x)$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log_e\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

Solved Problems:

Problem 1: Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ where $n \in \mathbb{Z}$.

Soln:

$$(\cosh x + \sinh x)^n = (\cos ix + \cancel{i}\sin ix)^n$$

$$= [\cos(ix) - i\sin(ix)]^n$$

$$= \cos n(ix) - i\sin n(ix)$$

$$= \cos i(nx) - i\sin i(nx)$$

$$= \cosh nx + \sinh(nx).$$

Problem 2: Prove that $\frac{1+\tanh x}{1-\tanh x} = \cosh 2x + \sinh 2x$.

Soln:

$$\frac{1+\tanh x}{1-\tanh x} = \frac{1+(\sinh x/\cosh x)}{1-(\sinh x/\cosh x)} = \frac{\frac{\cosh x+\sinh x}{\cosh x}}{\frac{\cosh x-\sinh x}{\cosh x}}$$

$$= \frac{\cosh x+\sinh x}{\cosh x} \times \frac{\cosh x}{\cosh x-\sinh x}$$

$$= \frac{\cosh x + \sinh x}{\cosh x - \sinh x} \times \frac{\cosh x + \sinh x}{\cosh x + \sinh x}$$

$$= \frac{\cosh^2 x + \sinh^2 x + 2\sinh x \cosh x}{\cosh^2 x - \sinh^2 x}$$

$$= \cosh 2x + \sinh 2x.$$

Problem 3: If $\tan(a+ib) = x+iy$ prove that $\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$

Soln:

$$x+iy = \tan(a+ib) = \frac{\sin(a+ib)}{\cos(a+ib)} \times \frac{\cos(a-ib)}{\cos(a-ib)}$$

$$= \frac{2\sin(a+ib)\sin(a-ib)}{2\cos(a+ib)\cos(a-ib)}$$

$$= \frac{\sin 2a + \sin(i2b)}{\cos 2a + \cos(i2b)} = \frac{\sin 2a + i\sin 2b}{\cos 2a + \cosh 2b}$$

$$x = \frac{\sin 2a}{\cos 2a + \cosh 2b}$$ and

$$y = \frac{\sinh 2b}{\cos 2a + \cosh 2b}$$

$$\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$$

Problem 4:

Prove that : If $x+iy = \tan(A+iB)$ prove that

$x^2+y^2+2x\cot 2A = 1.$

Soln:

$$x+iy = \tan(A+iB)$$

$$x-iy = \tan(A-iB)$$

$\tan 2A = \tan[(A+iB)+(A-iB)]$

$$\cot 2A = \frac{1}{\tan 2A} = \frac{1}{\tan[(A+iB)+(A-iB)]}$$

$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$

$$= \frac{1-\tan(A+iB)\tan(A-iB)}{\tan(A+iB)+\tan(A-iB)}$$

$(a^2-b^2)=(a+b)(a-b)$

$$= \frac{1-(x+iy)(x-iy)}{(x+iy)+(x-iy)} = \frac{1-(x^2+y^2)}{2x}$$

$$2x\cot 2A = 1-x^2-y^2$$

$$x^2+y^2+2x\cot 2A = 1.$$

Problem 5: If $x+iy = \sin(A+iB)$ prove that

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1.$$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

Soln:

$$x+iy = \sin(A+iB)$$

$$= \sin A \cos(iB) + \cos A \sin(iB)$$

$\sin(iB) = i\sinh B$

$$= \sin A \cosh B + i\cos A \sinh B$$

$$x = \sin A \cosh B \quad \text{and} \quad y = \cos A \sinh B$$

$x^2 = \sin^2 A \cosh^2 B \qquad y^2 = \cos^2 A \sinh^2 B$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1.$$

Problem 6: If $\cos(x+iy) = \cos\theta + i\sin\theta$ prove that

$\cos 2x + \cosh 2y = 2.$

Soln: $\cos\theta + i\sin\theta = \cos(x+iy)$

$$= \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y$$

$\cos\theta = \cos x \cosh y$ and $\sin\theta = -\sin x \sinh y$

Squaring and adding we get $1 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$$1 = \cos^2 x \cosh^2 y + (1-\cos^2 x)\sinh^2 y$$

$$1 = \cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y$$

$$1 = \cosh^2 x + \sinh^2 y$$

$$= \frac{1}{2}(1+\cos 2x) + \frac{1}{2}(\cosh 2y - 1)$$

$\cos 2x + \cosh 2y = 2.$

Problem 7: If $\sin(\theta + i\varphi) = \tan\alpha + i\sec\alpha$ prove that

★ $\cos 2\theta \cosh 2\varphi = 3.$

Soln:

$$\sin(\theta + i\varphi) = \tan\alpha + i\sec\alpha$$

$$\sin\theta \cos i\varphi + \cos\theta \sin i\varphi = \tan\alpha + i\sec\alpha$$

$$\sin\theta \cosh\varphi + \cos\theta\,(i\sinh\varphi) = \tan\alpha + i\sec\alpha$$

$\sin\theta \cosh\varphi = \tan\alpha$ and $\cos\theta \sinh\varphi = \sec\alpha$

$$1+\tan^2\alpha = \sec^2\alpha$$

$$1+\sin^2\theta\cosh^2\varphi = \cos^2\theta\sinh^2\varphi$$

$$1+(1-\cos^2\theta)\cosh^2\varphi = \cos^2\theta\sinh^2\varphi$$

$$1+\cosh^2\varphi - \cos^2\theta\cosh^2\varphi = \cos^2\theta\sinh^2\varphi$$

$$1+\cosh^2\varphi - \cos^2\theta(\cosh^2\varphi + \sinh^2\varphi) = 0$$

$\cosh 2\varphi = \cosh^2\varphi + \sinh^2\varphi$

$$1+\cosh^2\varphi - \cos^2\theta\cosh 2\varphi = 0$$ (By using recall-1)

$$1+\frac{1}{2}(1+\cosh 2\varphi) - \frac{1}{2}(1+\cos 2\theta)\cosh 2\varphi = 0$$

$$\frac{1}{2}\left[2+(1+\cosh 2\varphi) - (1+\cos 2\theta)\cosh 2\varphi\right] = 0$$

$$2+1+\cosh 2\varphi - (1+\cos 2\theta)\cosh 2\varphi = 0$$

$$3+\cosh 2\varphi - \cosh 2\varphi - \cos 2\theta\cosh 2\varphi = 0$$

$$3-\cos 2\theta\cdot\cosh 2\varphi = 0$$

$$\cos 2\theta\cosh 2\varphi = 3.$$

Problem 8: If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$ prove that

$$y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$$

Soln:

$$\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$$

$$\cos x\cos(iy) - \sin x\sin(iy) = r(\cos\alpha + i\sin\alpha)$$

$$\cos x\cosh y - \sin x(i\sinh y) = r(\cos\alpha + i\sin\alpha)$$

$$r\cos\alpha = \cos x\cosh y \quad \text{and} \quad r\sin\alpha = -\sin x\sinh y$$

$y = \tanh^{-1}(-\tan\alpha/\tan x)$ $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$= \frac{1}{2}\log\left[\frac{1-(\tan\alpha/\tan x)}{1+(\tan\alpha/\tan x)}\right] = \frac{1}{2}\log\left[\frac{\tan x - \tan\alpha}{\tan x + \tan\alpha}\right]$$

$$= \frac{1}{2}\log\left[\frac{\sin x\cos\alpha - \cos x\sin\alpha}{\sin x\cos\alpha + \cos x\sin\alpha}\right] = \frac{1}{2}\log\left[\frac{\sin(x-a)}{\sin(x+a)}\right]$$

Problem 9: If $\tan(x/2) = \tanh(x/2)$ prove that $\cos x \cosh x = 1$.

$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

soln: $\cosh x = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)} = \frac{1+\tan^2(x/2)}{1-\tan^2(x/2)}$

$\cos 2(x/2) = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)}$ $\cos 2(\theta/2) = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

$= \frac{1}{\cos x}$

$\cos x \cdot \cosh x = 1.$

★ Problem 10: prove $u = \log_e \tan\left(\frac{\pi}{4}+\theta/2\right)$ <u>if and only if</u> $\cosh u = \sec\theta$.

soln: Let $\cosh u = \sec\theta$

$\cosh^{-1}x = \log_e(x+\sqrt{x^2-$

$u = \cosh^{-1}(\sec\theta)$

$\tan^2\theta = \sec^2\theta - 1$

$$= \log_e\left[\sec\theta + \sqrt{\sec^2\theta - 1}\right] = \log_e(\sec\theta + \tan\theta) = \log_e\left(\frac{1}{\cos\theta} \right.$$

$$= \log_e\left(\frac{1+\sin\theta}{\cos\theta}\right) = \log_e\left[\frac{1+\left(\frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}\right)}{\frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}}\right]$$

$$= \log_e \frac{[1+\tan(\theta/2)]^2}{1-\tan^2(\theta/2)} = \log_e\left[\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right]$$

$$= \log_e\left[\frac{\tan(\pi/4)+\tan(\theta/2)}{1-\tan(\pi/4)\tan(\theta/2)}\right] = \log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

Conversely let $u = \log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$$e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \quad \text{Hence,} \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}$$

we take

$$\frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \frac{1+\tan(\theta/2)-[1-\tan(\theta/2)]}{1+\tan(\theta/2)+[1-\tan(\theta/2)]} = \frac{2\tan(\theta/2)}{2}$$

$\tanh x = \frac{\sinh x}{\cosh x}$

$$\tanh(u/2) = \tan(\theta/2) \longrightarrow (1)$$

$$\cosh u = \frac{1+\tanh^2(u/2)}{1-\tanh^2(u/2)} = \frac{1+\tan^2(\theta/2)}{1-\tan^2(\theta/2)}$$

$$\cosh u = \sec\theta.$$

Problem 11: Prove that

$$\cosh^7 x = (1/2^6)\left[\cosh 7x + 7\cosh 5x + 21\cosh 3x + 35\cosh x\right]$$

Soln: $\cosh^7 x = \left(\frac{e^x + e^{-x}}{2}\right)^7$

$$= \frac{1}{2^7}\left[e^{7x} + 7e^{5x} + 21e^{3x} + 35e^{x} + 35e^{-x} + 21e^{-3x} + 7e^{-5x} + e^{-7x}\right]$$

$$= \frac{1}{2^7}\left[(e^{7x} + e^{-7x}) + 7(e^{5x} + e^{-5x}) + 21(e^{3x} + e^{-3x}) + 35(e^{x} + e^{-x})\right]$$

$$= \frac{1}{2^7}\left[2\cosh 7x + 7(2\cosh 5x) + 21(2\cosh 3x) + 35(2\cosh x)\right]$$

$$\cosh^7 x = \frac{1}{2^6}\left[\cosh 7x + 7\cosh 5x + 21\cosh 3x + 35\cosh x\right]$$

Problem 12: Separate into real and imaginary parts

(i) $\tan^{-1}(x+iy)$

(ii) $\sin^{-1}(\cos\theta + i\sin\theta)$

(iii) $\sinh(\alpha + i\beta)$

(iv) $\tanh(1+i)$

Solution:

(i) Let $\tan^{-1}(x+iy) = A+iB$

$$\tan(A+iB) = x+iy \quad \text{--- (1)}$$

$$\tan(A-iB) = x-iy \quad \text{--- (2)}$$

(1)+(2) $\tan 2A = \tan\left(\overline{A+iB} + \overline{A-iB}\right)$

$$= \frac{\tan(A+iB) + \tan(A-iB)}{1 - \tan(A+iB)\tan(A-iB)}$$

$$= \frac{(x+iy) + (x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-(x^2+y^2)}$$

$$2A = \tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

The real part A is $\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-(x^2+y^2)}\right)$

① $-$ ② $\Rightarrow$ $\tan 2(iB) = \tan(\overline{A+iB} - \overline{A-iB})$

$$i\tanh 2B = \frac{\tan(A+iB) - \tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$= \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2iy}{1+x^2+y^2}$$

$$\tanh 2B = \frac{2y}{1+x^2+y^2}$$

$$2B = \tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

The imaginary part B is $\frac{1}{2}\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$.

(ii) Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

$$\cos\theta + i\sin\theta = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$$

$$\cos\theta = \sin x\cosh y \longrightarrow ①$$

$$\sin\theta = \cos x\sinh y \longrightarrow ②$$

Squaring and adding we get

$$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$$

$$1 = \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y$$

$$1 = \sin^2 x + \cancel{\sin^2 x\sinh^2 y} + \sinh^2 y - \cancel{\sin^2 x\sinh^2 y}$$

$$\sin^2 x + \sinh^2 y = 1$$

$$\sinh^2 y = 1 - \sin^2 x = \cos^2 x$$

$$\sinh y = \cos x \longrightarrow ③$$

From ② & ③ we get $\cos^2 x = \sin x$

$$\cos x = \sqrt{\sin x}$$

the real part is $\cos^{-1}(\sqrt{\sin\theta})$

From ③ & $\sinh y = \sqrt{\sin\theta}$

the imaginary part is $\sinh^{-1}(\sqrt{\sin\theta})$.

(iii) Let $x+iy = \sinh(\alpha + i\beta)$

$$x+iy = \frac{1}{i}\sin[i(\alpha+i\beta)] = -i\sin(i\alpha-\beta)$$

$$= -i[\sin(i\alpha)\cos\beta - \cos(i\alpha)\sin\beta]$$

$$= -i[i\sinh\alpha\cos\beta - \cosh\alpha\sin\beta]$$

$$= \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$$

The real part is $\sinh\alpha \cdot \cos\beta$

The imaginary part is $\cosh\alpha\sin\beta$

(iv) Let $\tanh(1+i) = x+iy$

$$x+iy = \frac{\sinh(1+i)}{\cosh(1+i)} = -i\frac{\sin i(1+i)}{\cos i(1+i)}$$

$$= -i\frac{\sin(i-1)}{\cos(i-1)} = -i\frac{2\sin(i-1)\cos(i+1)}{2\cos(i-1)\cos(i+1)}$$

$$= -i\frac{[\sin 2i - \sin 2]}{\cos 2i + \cos 2} = -i\frac{(i\sinh 2 - \sin 2)}{\cosh 2 + \cos 2}$$

$$= \frac{\sinh 2 + i\sin 2}{\cosh 2 + \cos 2}$$

Real part $= \dfrac{\sinh 2}{\cosh 2 + \cos 2}$ and Imaginary part $= \dfrac{\sin 2}{\cosh 2 + \cos 2}$

$\sin^2\theta = \pm \sin\alpha$

Soln: $\cos\alpha + i\sin\alpha = \cos(\theta + i\varphi)$

$= \cos\theta \cos(i\varphi) - \sin\theta . \sin(i\varphi)$

$\cos\alpha + i\sin\alpha = \cos\theta \cosh\varphi - i\sin\theta \sinh\varphi$

$\cos\alpha = \cos\theta \cosh\varphi \longrightarrow ①$

$\sin\alpha = -\sin\theta \sinh\varphi \longrightarrow ②$

From ① & ② $\cosh\varphi = \dfrac{\cos\alpha}{\cos\theta}$ and

$\sinh\varphi = \dfrac{-\sin\alpha}{\sin\theta}$

we take

$\cosh^2\varphi - \sinh^2\varphi = 1 \Rightarrow \dfrac{\cos^2\alpha}{\cos^2\theta} - \dfrac{\sin^2\alpha}{\sin^2\theta} = 1$

$\cos^2\alpha \sin^2\theta - \sin^2\alpha \cos^2\theta = \cos^2\theta \sin^2\theta$

$(1-\sin^2\alpha)\sin^2\theta - \sin^2\alpha(1-\sin^2\theta) = (1-\sin^2\theta)\sin^2\theta$

$\sin^2\theta - \cancel{\sin^2\theta \sin^2\alpha} - \sin^2\alpha + \cancel{\sin^2\alpha \sin^2\theta} = \sin^2\theta - \sin^4\theta$

$\sin^2\theta - \sin^2\alpha = \sin^2\theta - \sin^4\theta$

$-\sin^2\alpha = \cancel{\sin^2\theta} - \cancel{\sin^2\theta} - \sin^4\theta$

$\sin^4\theta = \sin^2\alpha$

$\sin^2\theta = \pm \sin\alpha$

Problem 14 : If $\tan(\theta + i\varphi) = \cos\alpha + i\sin\alpha$ prove that

(i) $\theta = \frac{1}{2} n\pi + \frac{1}{4}\pi$ (ii) $\varphi = \frac{1}{2} \log \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

Soln:

$$\tan(\theta + i\varphi) = \cos\alpha + i\sin\alpha$$

$$\tan(\theta - i\varphi) = \cos\alpha - i\sin\alpha$$

(i) now $2\theta = (\theta + i\varphi) + (\theta - i\varphi)$

$$\tan 2\theta = \tan\left[(\theta + i\varphi) + (\theta - i\varphi)\right]$$

$$= \frac{\tan(\theta + i\varphi) + \tan(\theta - i\varphi)}{1 - \tan(\theta + i\varphi)\tan(\theta - i\varphi)}$$

$$= \frac{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{1-1} = \frac{2\cos\alpha}{0}$$

$$\tan 2\theta = \infty = \tan\frac{\pi}{2}$$

anything / 0 = ∞

$$2\theta = n\pi + \frac{\pi}{2}$$

$\boxed{n\pi = 0}$

$$\theta = \frac{1}{2} n\pi + \frac{\pi}{4}$$

(ii) $\tan(2i\varphi) = \tan\left[(\theta + i\varphi) - (\theta - i\varphi)\right]$

$$i\tan 2\varphi = \frac{\tan(\theta + i\varphi) - \tan(\theta - i\varphi)}{1 + \tan(\theta + i\varphi)\tan(\theta - i\varphi)}$$

$\frac{\cancel{2} i \sin\alpha}{\cancel{2}}$

$$= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1 + (\cos^2\alpha + \sin^2\alpha)} = i\sin\alpha.$$

$\tanh i 2\varphi = i\sin\alpha$

$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$\tanh 2\varphi = \sin\alpha$$

$$2\varphi = \tanh^{-1}(\sin\alpha)$$

$$= \frac{1}{2}\log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)$$

$$= \frac{1}{2}\log\left[\frac{1+\left(\frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}\right)}{1-\left(\frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}\right)}\right]$$

$$= \log\left(\frac{1+\tan(\alpha/2)}{1-\tan(\alpha/2)}\right)$$

$$= \log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

$$\varphi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

Chapter - 3

Logarithm of A Complex number

Definition:

Let $z = r(\cos\theta + i\sin\theta)$ be a non-zero complex number we define $\boxed{\log z = \log r + i\theta.}$

Note : 1 Since the amplitude θ has infinite number of values, which differ by $2n\pi$ for some $n \in Z$. $\log z$ has infinite number of values which differ by $2in\pi$. If we take θ to be the principal value of the amplitude of z, the corresponding value of $\log z$ is called its principal value.

Hence if we have denote the general values of $\log z$ by $\text{Log } z$, we have

$z = r(\cos\theta + i\sin\theta)$

$x + iy = r\cos\theta + ir\sin\theta$

$x = r\cos\theta \quad y = r\sin\theta$

$$\text{Log } z = \log z + i(2n\pi) \text{ where } n \in Z$$

$x^2 + y^2 = r^2(\sin^2\theta + \cos^2\theta)$

$r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$

$$\boxed{\text{Log } z = \log r + i(\theta + 2n\pi) \text{ where } n \in Z.}$$

Note 2: Let $z = x + iy = r(\cos\theta + i\sin\theta)$

$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

Hence $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

$\theta = \tan^{-1}(y/x)$

$$\log z = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$$

Note 3: If $z = x$ is real and positive, then $\mathrm{Log}\, z = \log x + i$ where $n \in \mathbb{Z}$. Hence one value of $\mathrm{Log}\, z$ is a positive real number and all other values are non-real

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Theorem: $\mathrm{Log}\, z_1 z_2 = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$.

proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2 [\cos\theta_1 + i\sin\theta_1 + \cos\theta_2 + i\sin\theta_2]$$

$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$\mathrm{Log}\, z = \log r + i(\theta + 2n\pi)$, $n \in \mathbb{Z}$

$$\mathrm{Log}\, z_1 z_2 = \log r_1 r_2 + i(\theta_1 + \theta_2 + 2n\pi), n \in \mathbb{Z}$$

$$= \log r_1 + \log r_2 + i(\theta_1 + 2n\pi) + i(\theta_2 + 2m\pi)$$

where, $m, n \in \mathbb{Z}$.

$$= [\log r_1 + i(\theta_1 + 2n\pi)] + [\log r_2 + i(\theta_2 + 2m\pi)]$$

$$= \log z_1 + \log z_2.$$

Definition: If z and ω are any two complex numbers then we define $\boxed{z^{\omega} = e^{\omega \log z}}$

Note: Since $\mathrm{Log}\, z$ has infinite no of value for z, z^{ω} also has infinite numbers values.

problem 1 : Prove that $Log\, i = i(4n+1)(\pi/2)$.

soln:

$Log\, z = log\, r + i(\theta + 2n\pi)$

$$Log\, i = log\, 1 + i\left[(\pi/2) + 2n\pi\right]$$

$$= i\left[(\pi/2) + 2n\pi\right]$$

$$= i(4n+1)(\pi/2)$$

problem 2 : Find (i) $Log(1+i)$ (ii) $Log(-e)$

soln:

(i) $1+i = \sqrt{2}\left[\cos(\pi/4) + i\sin(\pi/4)\right]$

$$Log(1+i) = log\sqrt{2} + i(\pi/4) + 2n\pi i.$$

$$z = 1+i \quad \Rightarrow z = x+iy$$

$$x = 1 \quad y = 1$$

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}(y/x) = \tan^{-1}(1/1) = \pi/4$$

$$Log\, z = log\, r + i(\theta + 2n\pi),\ n \in z.$$

$$Log(1+i) = log\sqrt{2} + i\left(\pi/4 + 2n\pi\right)$$

$$= log\sqrt{2} + i\,\pi/4 + i2n\pi$$

$$= log\, 2^{1/2} + i\left(\frac{8n\pi + \pi}{4}\right)$$

$$Log(1+i) = \frac{1}{2}\, log\, 2 + i(8n+1)(\pi/4)$$

(ii)

$$z = r(\cos\theta + i\sin\theta)$$

$$-e = e(\cos\pi + i\sin\pi)$$

$$Log\, z = log\, r + i(\theta + 2n\pi)$$

$$Log(-e) = log\, e + i(\pi + 2n\pi)$$

$$= 1 + i(2n+1)\pi.$$

Problem 3: Prove that

$$\operatorname{Log}\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{\operatorname{cosec}(\theta/2)}{2}\right) + i\left(2n\pi + \pi/2 - \theta/2\right)$$

Soln:

$$\frac{1}{1-e^{i\theta}} = (1-e^{i\theta})^{-1}$$

$$= \left[1-(\cos\theta + i\sin\theta)\right]^{-1}$$

$$= \left[(1-\cos\theta) - i\sin\theta\right]^{-1}$$

$$= \left[(1-\cos\theta) - i\sin\theta\right]^{-1}$$

$$= \left[2\sin^2\theta/2 - i\,2\sin\theta/2\cos\theta/2\right]^{-1}$$

$$= (2\sin\theta/2)^{-1}(\sin\theta/2 - i\cos\theta/2)^{-1}$$

$$= \frac{1}{2\sin\theta/2}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{-1}$$

$$= \frac{\operatorname{cosec}\theta/2}{2}\left[\cos(\pi/2-\theta/2) + i\sin(\pi/2-\theta/2)\right]$$

$$\operatorname{Log} z = \log r + i(\theta + 2n\pi),\ n \in \mathbb{Z}.$$

$$\operatorname{Log}\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{\operatorname{cosec}\theta/2}{2}\right) + i\left(\pi/2 - \theta/2\right) + i(2n\pi)$$

$$= \log\left(\frac{\operatorname{cosec}\theta/2}{2}\right) + i\left(2n\pi + \pi/2 - \theta/2\right)$$

Problem 4: Prove that $i^i = e^{-(4n+1)(\pi/2)}$.

Soln:

$$z^\omega = e^{\omega \operatorname{Log} z}$$

$$i^i = e^{i \operatorname{Log} i} = e^{i(i(4n+1)(\pi/2))}$$

$$= e^{i^2(4n+1)(\pi/2)}$$

$$i^i = e^{-(4n+1)(\pi/2)}$$

Problem 5: If $i^{a+ib} = a+ib$ prove that $a^2+b^2 = e^{-(4n+1)\pi b}$.

<u>Soln:</u>

$$i^{a+ib} = a+ib \longrightarrow (1)$$

$$i^{a+ib} = e^{(a+ib)\operatorname{Log} i} \qquad z^w = e^{w \log z}$$

$$= e^{(a+ib)\, i(4n+1)(\pi/2)}$$

$$= e^{(ai-b)(4n+1)(\pi/2)}$$

$$= e^{ia(4n+1)(\pi/2)} \cdot e^{-b(4n+1)(\pi/2)}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$= e^{-b(4n+1)(\pi/2)} \left[\cos\theta + i\sin\theta\right]$$

where $\theta = a(4n+1)(\pi/2)$

by (1), $a+ib = e^{-b(4n+1)(\pi/2)} \left[\cos\theta + i\sin\theta\right]$

Equating real and imaginary parts,

$$a = e^{-b(4n+1)(\pi/2)} \cdot \cos\theta$$

$$b = e^{-b(4n+1)(\pi/2)} \cdot \sin\theta$$

$$a^2+b^2 = e^{-b(4n+1)\pi} = e^{-(4n+1)\pi b}$$

Exercises:

1) Prove ~~that~~ the following :

(i) $\text{Log } 1 = i 2n\pi$.

Soln:

$$z = x+iy = 1+i0$$

$$x = 1 \quad y = 0$$

$$r = \sqrt{x^2+y^2} = \sqrt{1} = 1$$

$$\theta = \tan^{-1}(y/x) = \tan^{-1}(0/1) = \tan^{-1} 0 = 0$$

$$\text{Log } z = \log r + i(\theta + 2n\pi), n \in Z.$$

$$\text{Log } 1 = \log 1 + i(0 + 2n\pi)$$

$$\text{Log } 1 = i 2n\pi.$$

(ii) $\text{Log}(-1) = i(2n+1)\pi$

$$z = x+iy = -1+i0$$

$$x = -1 \quad y = 0$$

$$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{-1}\right) = \pi$$

$$\text{Log } z = \log r + i(\theta + 2n\pi), n \in Z$$

$$\text{Log}(-1) = \log 1 + i(\pi + 2n\pi)$$

$$\text{Log}(-1) = i(1+2n)\pi$$